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Consistent XVA Metrics Part I: Single-currency

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Abstract

We present a consistent framework for computing shareholder and firm values of derivative portfolios in the presence of collateral, counterparty risk and funding costs in a single currency economy with stochastic interest rates and spot assets with local volatility. The follow-up paper Kjaer [12] extends this setup to a multi-currency economy and the resulting valuation adjustments have been implemented in the forthcoming Bloomberg MARS XVA product.

Keywords. Shareholder and firm values, Valuation adjustments, counterparty risk, collateral, CSA discounting, Bloomberg MARS XVA.

DISCLAIMER

Contents

1 Executive summary

We provide consistent calculations for shareholder and firm value adjustments ("The XVA metrics") for a derivative portfolio between a bank and a counterparty in the presence of counterparty risk, funding costs and collateral support annexes ("CSA") in a single-currency economy. All calculations are from the perspective of "the bank".

Shareholder, firm and reference values

The *shareholder value* of a derivative portfolio depends on the bank funding strategy and spreads. It does not include the value of payments after own default. The *firm value* is the combined value of the bank to the shareholders and bondholders. It includes the value of payments after own default but excludes payments between shareholders and bondholders and is thus independent of the funding strategy.

Valuation adjustments are calculated with respect to a *risk-free reference value* which uses OIS discounting and is abbreviated V_{OIS}. We also encounter the CSA discounted value V_{CSA} where the collateral rate is given as a fixed spread over OIS.

The XVA metrics

The *shareholder value* $\hat{V}$ can be decomposed as:

$$\hat{V} = V_{\mathrm{OIS}} + COLVA + FCVA + FVA + MVA$$

with *shareholder value metrics*

- $COLVA$: The *collateral valuation adjustment* given as $V_{\mathrm{CSA}} - V_{\mathrm{OIS}}$.
- $FCVA$: The *funding curve discounted credit valuation adjustment* which is the cost of hedging counterparty default risk.
- FVA: The *funding valuation adjustment* which is the cost or gain of funding or investing the net variation margin collateral due to the unsecured funding rate being different from the CSA collateral rate.
- MVA: The *margin valuation adjustment* which is the cost of raising unsecured funding for the initial margin whilst only earning the CSA rate.

The *firm value* $\hat{V}_{\mathrm{FV}}$ can be decomposed as:

$$\hat{V}_{\mathrm{FV}} = V_{\mathrm{OIS}} + COLVA + FTDCVA + FTDDVA + FVVMVA + FVIMVA$$

with *firm value metrics*

- $FTDCVA$: The *first-to-default CVA* is the cost of counterparty default in scenarios when the bank has not defaulted.

- $FTDDVA$: The *first-to-default DVA* is the gain to the bank bond holders of not having to pay all liabilities at own default in scenarios when the counterparty has not defaulted.
- $FVVMVA$: The *firm value variation margin value adjustment* is the cost or gain due to the variation margin collateral rate being different from the OIS reference rate.
- $FVIMVA$: The *firm value initial margin value adjustment* is the cost or gain due to the initial margin collateral rate being different from the OIS reference rate.

Credit, funding and CSA curves

All XVA metrics need the following market data:

- Counterparty credit curve.
- OIS discounting curve.
- Variation and initial margin collateral spreads.

Shareholder value metrics only:

- Bank unsecured (i.e. risky) discount curve (the same rate is used for funding and investing).

Firm value metrics only:

- Bank credit curve.

Other assumptions

1. Frictionless continuous time and amount trading.
2. All trade cash flows, collateral and hedge assets are denominated in a single (domestic) currency.
3. All hedge assets are traded on a collateralised (or repo) basis.
4. The market risk factors are independent of J_B and J_C (e.g. no jump in S or r at counterparty default).
5. Deterministic funding, Libor, collateral and credit spreads.
6. Single bond funding strategy used.
7. Full re-hypothecation of variation margin collateral.
8. Initial margin collateral is held by a third party custodian who pays the interest on it.
9. No basis spreads between bank debt of different seniorities.
10. Deterministic recovery rates for bank and counterparty debt and derivatives (these recovery rates can be different).

2 Introduction

Many models for computing valuation adjustments have been developed since the start financial crisis, for example Piterbarg [13], Burgard and Kjaer [6], [7],[8], [9], Pallavicini et. al. [4] and Albanese and Andersen [1]. Many of the more important results from these papers are discussed in Green [10]. Credit, funding and other valuation adjustments (henceforth abbreviated XVA) are now routinely included in dealer pricing quotes and found in the financial statements of many large institutions. The aim of this paper is to present a model for consistent valuation adjustments in a single currency economy. We use the same semi-replication approach as in Burgard and Kjaer [8] since we feel it is very intuitive and offers the necessary detailed control over the precise funding of different types of cash flows, including the hedging of the valuation adjustments themselves. We extend Burgard and Kjaer [8] in four ways. First the asset dynamics is rich enough to include the models most commonly used in practice for XVA calculations, such as the Hull-White one-factor model for interest rates and local volatility for spot assets. Second the collateral modelling is more careful and we distinguish between re-hypothecable variation margin and initial margin held by a third party. It is this separation that causes the margin value adjustment, or MVA. Three we distinguish between OIS and CSA discounting which gives rise to the collateral valuation adjustment, or $COLVA$. Four, and finally, we derive valuation adjustments not only for the shareholder value but also for the firm value. This is consistent with the approach taken in Albanese and Andersen [1], Andersen and Duffie [2] and Burgard and Kjaer [9]. In a subsequent paper we plan to extend this setup further to multi-currency portfolios, trades, collateral and counterparties.

This paper is organised as follows: The assets and accounts of the single currency economy are introduced in Section 3 before we use semi-replication do derive the general valuation PDEs for the shareholder and firm values in Section 4. To keep the notation reasonably simple we will initially restrict ourselves to portfolios with a single netting set and credit support annex. In Section 5 we choose a specific funding strategy with a single bond and derive the resulting shareholder and firm valuation adjustment metric formulas. These formulas are then interpreted in Section 6 followed by a discussion about the model assumptions in Section 7. We conclude in Section 8 and provide technical details and an extension to multiple netting sets and credit support annexes in Appendix A to D.

3 Hedging assets, accounts and rates

In this paper we consider a portfolio of derivative trades between a bank B and a counterparty C in a single currency economy. To reduce the complexity of the notation we assume that all trades belong to a single netting set and CSA. The extension to multiple netting sets and CSAs is straightforward and discussed in Appendix C. The assets, accounts and rates are listed in Table 1.

Z_r^T	Default risk-free zero-coupon (maturing at T) bond price.
β_Z, r	Repo-account and rate secured against Z_r^T.
r_L	Continuously compounded instantaneous Libor rate
S	Spot asset price.
β_S, γ_S	Repo-account and rate secured against S.
P_C, r_C	Counterparty overnight bond price and rate.
β_C, γ_C	Repo-account and rate secured against P_C.
$P_{F,j}, r_{F,j}$	Bank un-secured overnight bond price and rate for seniority j.
β_ϕ, r_ϕ	Variation margin collateral account and rate.
β_ψ, r_ψ	Initial margin collateral account and rate.

Table 1: Assets, accounts and rates of the single currency economy. The bank bond index j is ordered by increasing seniority. The account values $\beta_Z, \beta_S, \beta_C, \beta_\phi, \beta_\psi$ are per unit of account.

All the assets in Table 1 are traded on a repo (or collateralised) basis and the associated account values are per unit of account. In particular we follow Piterbarg [14] and let the "risk-free" rate r of our economy be defined as the repo (or collateral) rate on the default risk-free bond Z. We use the notation r rather than γ_Z partly for sentimental reasons and partly because the former looks more aesthetic in a stochastic differential equation. As in Burgard and Kjaer [8],[9] the bank has multiple overnight bonds $P_{F,j}$ of different seniorities (and recovery rates) $R_{F,j}$ for funding purposes. Throughout this paper we use the rate ξ discount factor $D_\xi(t,T) = \exp\left(-\int_t^T \xi(u)du\right)$.

We next assume that the dynamics of the assets, accounts and rates in Table 1 under the real world probability measure are given by

$$
\begin{aligned}
dS(t) &= \mu_S(t,S(t))S(t)dt + \sigma_S(t,S(t))S(t)dW_S(t) \\
dr(t) &= \mu_r(t,r(t))dt + \sigma_r(t,r(t))dW_r(t) \\
dP_C(t) &= r_C(t)P_C(t^-)dt - (1-R_C)P_C(t^-)dJ_C(t) \\
dP_{F,j}(t) &= r_{F,j}(t)P_{F,j}(t^-)dt - (1-R_{F,j})P_{F,j}(t^-)dJ_B(t) \\
d\beta_S(t) &= (\gamma_S(t) - q_S(t))\beta_S(t)dt \\
d\beta_Z(t) &= r(t)\beta_Z(t)dt \\
d\beta_C(t) &= \gamma_C(t)\beta_C(t)dt \\
d\beta_\phi(t) &= r_\phi(t)\beta_\phi(t)dt \\
d\beta_\psi(t) &= r_\psi(t)\beta_\psi(t)dt.
\end{aligned}
\tag{1}
$$

Here W_r and W_S are Wiener-processes with correlation ρ and J_B and J_C are independent Poisson processes. The recovery rates $0 \le R_C \le 1$, $0 \le R_{F,0} < R_{F,1} < \ldots \le 1$ are constant and the functions $\mu_S(t,s), \sigma_S(t,s), \mu_r(t,r), \sigma_r(t,r)$ satisfy certain technical conditions to guarantee strong solutions of the stochastic differential equations for S and r. We furthermore assume a deterministic spot asset dividend yield q_S and that all other rates are deterministic spreads over r such that

$$\begin{aligned} r_L(t) &= r(t) + s_L(t) \\ r_C(t) &= r(t) + s_C(t) \\ r_{F,j}(t) &= r(t) + s_{F,j}(t) \\ \gamma_C(t) &= r(t) + s_{\beta,C}(t) \\ \gamma_S(t) &= r(t) + s_{\beta,S}(t) \\ r_\phi(t) &= r(t) + s_\phi(t) \\ r_\psi(t) &= r(t) + s_\psi(t) \end{aligned} \tag{2}$$

In the case of zero basis between bank bonds of different seniority it is straightforward to show that

$$s_{F,j} = (1 - R_{F,j})\lambda_B \tag{3}$$

where λ_B is the deterministic spread of a (potentially hypothetical) bank zero recovery bond. The construction of a funding curve $r_{F,j}$ is discussed in Appendix D. Analogously the spread of a counterparty zero recovery bond is given by $\lambda_C := \frac{r_C - \gamma_C}{1 - R_C}$ which is deterministic.

Following standard short rate bond-price modelling for one-factor models (see e.g. Brigo and Mercurio [5]) we assume that $Z_r^T(t) = Z_r^T(t, r(t))$ so Itô's Lemma yields that

$$dZ_r^T(t) = \left(\frac{\partial Z_r^T}{\partial t}(t, r(t)) + \frac{1}{2}\sigma_r^2(t, r(t)) \frac{\partial^2 Z_r^T}{\partial r^2}(t, r(t)) \right) dt + \frac{\partial Z_r^T}{\partial r}(t, r(t)) dr(t) \tag{4}$$

which shows that this bond can be used to hedge interest rate risk. Next we introduce the *market price of interest rate risk* $\nu_r(t, r)$ given by

$$\nu_r(t, r) := \frac{\frac{\partial Z_r^T}{\partial t} + \frac{1}{2}\sigma_r^2(t, r)\frac{\partial^2 Z_r^T}{\partial r^2} + \mu_r(t, r)\frac{\partial Z_r^T}{\partial r} - rZ_r^T}{\sigma_r(t, r)\frac{\partial Z_r^T}{\partial r}}$$

and let $a_r(t, r) := \mu_r(t, r) - \nu_r(t, r)\sigma_r(t, r)$. The bond dynamics (4) can now be rewritten as

$$dZ_r^T(t) = r(t)Z_r^T(t)dt + \frac{\partial Z_r^T}{\partial r}(t, r(t)) \left(dr(t) - a_r(t, r(t))dt \right). \tag{5}$$

In practise we would typically specify some a-priori forms of $a_r(t, r)$ and $\sigma_r(t, r)$ to yield the Vasicek, Hull-White, CIR or other one-factor short rate model, and then calibrate it to e.g. market discount factors, caps and swaptions.

As the instantaneous Libor spread s_L is deterministic we can compute forward Libor rates with tenor τ as $L(t, T, T + \tau) = \left(D_{s_L}(T, T + \tau)\frac{Z_r^T(t)}{Z_r^{T+\tau}(t)} - 1 \right) \times \frac{1}{\tau}$. Consequently $L(t, T, T + \tau)$ is a function of $r(t)$.

For the remainder of this paper we suppress the explicit dependence on t to improve the clarity of the exposition and write $P_{F,j}^- := P_{F,j}(t^-)$.

4 Valuation by semi-replication

We consider a derivative portfolio whose trades pay the total amount $H(r(T), S(T))$ on the same date T provided that the bank and the counterparty are both alive. The payoff is expressed in terms of $r(T)$ as the Libor rate $L(T, T', T'+\tau)$ can be computed from $r(T)$. To allow more complex portfolio events like multiple European style trade cashflows on different dates and spot asset or Libor fixings prior to T would be straightforward extensions but would make the notation more complex.

As in Burgard and Kjaer [8], [9] we let $\hat{V} = \hat{V}(t, r, S, J_B, J_C)$ denote the total value to the bank of the portfolio including funding, collateral and counterparty risk. Here $\hat{V} \geq 0$ represents an asset to the bank. Like in Kjaer [11] we consider generic boundary conditions at default of the bank or the counterparty at time t given by $\hat{V}(t, r, S, 1, 0) := g_B(t)$ and $\hat{V}(t, r, S, 0, 1) := g_C(t)$, respectively. These boundary conditions represent the present value of the portfolio immediately after default and can represent contractual features such as standard ISDA closeouts, set-offs, extinguishers, netting (or absence thereof) and collateral. As discussed in Andersen, Pykhtin and Sokol [3] the modelling of closeout is complicated further by the margin period of risk during which some, but not all cash flows are made. Given the complexity of the topic we use general g_B and g_C in this paper.

4.1 Semi-replication

Still following Burgard and Kjaer [8], [9] we consider the bank balance sheet consisting of a derivative book with value $\hat{V}$ and a hedging and funding portfolio Π given by

$$\Pi = \delta_S S + \delta_Z Z_r^T + \delta_C P_C + \sum_j \delta_{F,j} P_{F,j} + \alpha_S \beta_S + \alpha_C \beta_C + \alpha_Z \beta_Z + \alpha_\phi \beta_\phi + \alpha_{\psi,B} \beta_\psi. \tag{6}$$

Here a positive weight means the bank is long the asset or account. In particular $\alpha_\phi \geq 0$ means the bank has posted variation margin collateral with the counterparty and the total collateral balance ϕ satisfies $\phi = \alpha_\phi \beta_\phi$. Analogously we let $\psi_B = \alpha_{\psi,B} \beta_\psi$ with $\alpha_{\psi,B} \geq 0$ and $\psi_C = \alpha_{\psi,C} \beta_\psi$ with $\alpha_{\psi,C} \leq 0$ denote the amounts of initial margin posted by the bank and the counterparty, respectively. The initial margin amount ψ_C does not feature in Equation (6) since it is assumed to be held by a third party custodian, who is also responsible for paying the interest on this balance. Thus the bank funded collateral $\phi + \psi_B$ is given by the sum of the variation margin plus the initial margin posted by the bank. The rules for computing variation and initial margin can be quite complex and are not discussed in this paper.

The hedge assets and their repo accounts satisfy the the relations

$$\begin{aligned}
\delta_S S + \alpha_S \beta_S &= 0 \\
\delta_C P_C + \alpha_C \beta_C &= 0 \\
\delta_Z Z_r + \alpha_Z \beta_Z &= 0.
\end{aligned} \tag{7}$$

In order to be replicating we require also require that $\hat{V} + \Pi = 0$ holds except possibly at bank default, which in turn implies that the funding weights $\delta_{F,j}$ must be chosen by the bank such that they satisfy the *funding constraint*

$$\hat{V} + \sum_j \delta_{F,j} P_{F,j} + \phi + \psi_B = 0. \tag{8}$$

Using Itô's lemma, the boundary conditions g_B and g_C, the no-basis condition (3) and choosing the hedge ratios δ_S, δ_Z and δ_C such that the market and counterparty default risks are hedged out gives the balance sheet dynamics

$$\begin{aligned} d(\hat{V} + \Pi) = &\Big\{ \frac{\partial \hat{V}}{\partial t} + \mathcal{A}\hat{V} + \lambda_B(g_B - \hat{V}) + \lambda_C(g_C - \hat{V}) - r(\hat{V} + \phi + \psi_B) + r_\phi \phi + r_\psi \psi_B \Big\} dt \\ &+ \epsilon_h (dJ_B - \lambda_B dt). \end{aligned} \tag{9}$$

with parabolic operator $\mathcal{A}$ given by

$$\begin{aligned} \mathcal{A} = &\frac{1}{2}\sigma_S^2(t,S)S^2 \frac{\partial^2}{\partial S^2} + \frac{1}{2}\sigma_r^2(t,r)\frac{\partial^2}{\partial r^2} + \rho\sigma_r(t,r)\sigma_S(t,S)S\frac{\partial^2}{\partial S \partial r} \\ &+ (\gamma_S - q_S)S\frac{\partial}{\partial S} + a_r(t,r)\frac{\partial}{\partial r}. \end{aligned}$$

and hedge error at own default $\epsilon_h = g_B - \hat{V} - \sum_j (1 - R_{F,j})\delta_{F,j} P^-_{F,j}$. The full proof is given in Appendix A.

4.2 Shareholder value

It can be seen from Equation (9) that the balance sheet is risk-free as long as the bank is alive. At bank default the jump term $\epsilon_h dJ_B$ gives rise to a wind or shortfall to the bondholders. When alive, the shareholders of the bank will transfer a cost/gain of $\lambda_B \epsilon_h$ per unit of time to the bond-holders. From the shareholder's perspective we want to know the initial cost of setting up a self-financing portfolio that replicates the final payoff H in all scenarios but possibly bank default. This cost is obtained by setting all the $dt-$terms in (9) to zero, leading to the PDE

$$\begin{aligned} \frac{\partial \hat{V}}{\partial t} + \mathcal{A}\hat{V} - (r + \lambda_B + \lambda_C)\hat{V} &= -\lambda_B g_B - \lambda_C g_C - (r_\phi - r)\phi - (r_\psi - r)\psi_B + \lambda_B \epsilon_h \\ \hat{V}(T, r, S, 0, 0) &= H(r, S) \end{aligned} \tag{10}$$

subject to the funding constraint (8). We define the *shareholder value* of the derivative portfolio to the the solution $\hat{V}$ to (10). It is clear that $\hat{V}$ depends on the funding strategy $\delta_{F,j}$ but critically not on g_B. This is expected as the shareholder value should in general not depend on a payment that happens post default. The only exception is if the funding strategy depends explicitly on g_B, as is the case in the perfect replication strategy of Burgard and Kjaer [8] where the funding weights are chosen such that $\epsilon_h = 0$.

Albanese et. al. [1] use accounting principles rather than semi-replication to define a shareholder value of a derivatives book in the presence of counterparty risk, funding and collateral. Burgard and Kjaer [9] prove that there exists a particular funding strategy such that the resulting $\hat{V}$ coincides with the definition in Albanese et. al. [1]. Our definition is more general as it is meaningful for all funding strategies satisfying the funding constraint (8).

Albanese et. al. [1], Andersen and Duffie [2] and Burgard and Kjaer [9] all argue that the shareholder value should be used for decision making and transfer pricing. Any new derivative must be charged at least the incremental shareholder value $\Delta\hat{V}$ to ensure the shareholders are not worse off.

4.3 Firm value

Albanese et. al. [1], Andersen and Duffie [2] and Burgard and Kjaer [9] all define the *firm value* $\hat{V}_{\mathrm{FV}}$ as the combined value of the derivative book to the shareholders and bondholders. It represents the cost of taking over the entire part of the balance sheet, including debt, that is associated with the derivative portfolio. From this definition we have to exclude the martingale $\epsilon_h(dJ_B - \lambda_B dt)$ part from the balance sheet dynamics (9) as it represents payments between the shareholders and bondholders. In other words, $\hat{V}_{\mathrm{FV}}$ satisfies (10) with $\epsilon_h = 0$. From this we see that the condition $\psi_B = -\psi_C$ is necessary for the firm value to be symmetric between the bank and the counterparty. As expected the firm value is independent of the funding strategy followed by the bank.

4.4 Risk neutral dynamics

By the Feynman-Kac theorem the solution to (10) can be expressed in terms of a conditional expectation $\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot|S(t) = s, r(t) = r]$ with respect to a probability measure $\mathbb{Q}$ under which $J_B(t)$ and $J_C(t)$ are independent Poisson processes with intensities $\lambda_B(t)$ and $\lambda_C(t)$ and

$$\begin{aligned} dS(t) &= (\gamma_S(t) - q_S(t))S(t)dt + \sigma_S(t, S(t))S(t)dW_S^{\mathbb{Q}}(t) \\ dr(t) &= a_r(t, r(t))dt + \sigma_r(t, r(t))dW_r^{\mathbb{Q}}(t) \end{aligned} \tag{11}$$

with $W_S^{\mathbb{Q}}(t)$ and $W_r^{\mathbb{Q}}(t)$ being $\mathbb{Q}$-Wiener processes with correlation ρ. By standard short rate theory, the zero coupon bond prices in this model are given by $Z_r^T(t) = \mathbb{E}_t[D_r(t, T)]$ which can be computed (semi)-analytically for the short rate models mentioned above.

4.5 Reference values

Before proceeding with the valuation adjustments in Section 5 we need to formalise the reference value we calculate the adjustments on top of such that the total equals $\hat{V}$ or $\hat{V}_{\mathrm{FV}}$. First we let ξ be an arbitrary rate such that $s_\xi(t) := \xi(t) - r(t)$ is deterministic. We then define the rate ξ-discounted value V_ξ to be the Feynman-Kac solution $V_\xi(t, r, S) = \mathbb{E}_t[D_\xi(t, T)H(r(T), S(T))]$ to the partial differential equation

$$\begin{aligned} &\frac{\partial V_\xi}{\partial t} + \mathcal{A}V_\xi = \xi V_\xi \\ &V_\xi(T, r, S) = H(r, S). \end{aligned} \tag{12}$$

In particular we will consider the two cases $\xi(t) = r(t)$ and $\xi(t) = r_\phi(t)$ where we denote the respective solutions by V_r and V_ϕ and refer to them as the OIS and CSA discounted values, respectively. The general shareholder value PDE (10) can be reduced to (12) with $\xi(t) = r_\phi(t)$ provided that $\psi_B = 0$, $\phi = -\hat{V}$, and $g_C = \hat{V}$. These conditions combined with the funding constraint (8) imply that $\sum_j \delta_{F,j} P_{F,j} = 0$ so $\hat{V} = V_\phi$ which is expected as there is no counterparty risk or unsecured funding requirements. In Section 6 we will give an example of a boundary condition g_C such that $g_C = \hat{V}$ is true. If furthermore $g_B = \hat{V}$ then the firm value satisfies $\hat{V}_{\text{FV}} = V_\phi$ as well.

5 Funding strategies and valuation adjustments

As seen in the previous section, and discussed at length in Burgard and Kjaer [8],[9], the shareholder value depends on the funding strategy deployed and so far in this paper the theory has been developed for an arbitrary funding strategy. Hereafter we focus our attention on the single bond funding strategy from Burgard and Kjaer [8] which in essence takes the model in Piterbarg [13] and adds counterparty risk, including own default risk, and initial margin. To simplify the notation the seniority index j is dropped from the notation from now on. This strategy features a symmetric borrowing and lending rate r_F since all borrowing is done by issuing P_F and all excess cash is invested by repurchasing P_F. Here the word "re-purchase" should be interpreted as reducing the short term borrowing requirements by not rolling over debt. This implicitly assumes the bank as a whole is a net borrower and that excess cash from the derivatives operations can be recycled elsewhere by the Treasury (e.g. by filling capital buffers). In Burgard and Kjaer [9] it is shown that the valuation adjustments under this strategy are additive across credit support annexes and netting sets, which in retrospect justifies why we only consider one counterparty in our economy. Under this strategy the funding constraint (8) yields

$$\delta_F P_F = -\hat{V} - \phi - \psi_B. \tag{13}$$

The shareholder value defined in Albanese and Andersen [1] corresponds to a different asymmetric funding strategy as discussed in Burgard and Kjaer [9]. Here a net borrowing requirement is funded at the rate r_F by issuing P_F-bonds, whereas a net cash surplus must be invested by purchasing Z_r-bonds earning the rate r (via a reversed repo-agreement). The net cash requirement or surplus is computed by the Treasury at bank level and covers all counterparties in the bank. This implies that the shareholder value is no longer additive per netting set but must be computed at bank level, which could be a formidable task from a computational and memory point of view. In practise the two strategies are identical unless the derivative operations as well as the bank as a whole have both generated cash surpluses at the same point in time. Only then would the bank Treasury not be able to re-cycle the derivatives cash surplus within the bank itself.

Having fixed the funding strategy we decompose the shareholder and firm values as

$$\begin{aligned}\hat{V} &= V_\phi + U \\ &= V_r + (V_\phi - V_r) + U \\ &:= V_r + COLVA + U \\ \hat{V}_{\mathrm{FV}} &= V_r + COLVA + U_{\mathrm{FV}}.\end{aligned} \tag{14}$$

The *Collateral Valuation Adjustment*, or $COLVA$, is the difference between the OIS and CSA discounted values. From the PDE (12) satisfied by V_ϕ and V_r it can be shown that

$$COLVA = -\mathbb{E}_t\left[\int_t^T (r_\phi(u) - r(u))D_r(t,u)V_\phi(u)du\right] \tag{15}$$

where the expectation is taken with respect to the measure Q introduced in Section 4.4. We interpret this integral as a basis swap in Section 6. For counterparties without CSA we set $V_\phi := V_r$ so $COLVA = 0$ in this case.

5.1 Shareholder value adjustment metrics

In Appendix B we prove that $U = FCVA + FVA + MVA$ with the *funding curve discounted credit, funding and margin* value adjustments given by

$$FCVA = -\mathbb{E}_t\left[\int_t^T \lambda_C(u)D_{r_F+\lambda_C}(t,u)(V_\phi(u) - g_C(u))du\right] \tag{16}$$

$$FVA = -\mathbb{E}_t\left[\int_t^T (r_F(u) - r_\phi(u))D_{r_F+\lambda_C}(t,u)(V_\phi(u) + \phi(u))du\right] \tag{17}$$

$$MVA = -\mathbb{E}_t\left[\int_t^T (r_F(u) - r_\psi(u))D_{r_F+\lambda_C}(t,u)\psi_B(u)du\right]. \tag{18}$$

In section 6 we provide interpretations for these valuation adjustments.

5.2 Firm value adjustment metrics

In Appendix B we prove that $U_{\mathrm{FV}} = FTDCVA + FTDDVA + FVVMVA + FVIMVA$ with the *first-to-default credit, first-to-default debit, firm value variation margin and firm value initial margin* value adjustments given by

$$\begin{aligned} FTDCVA(&= -\mathbb{E}_t\left[\int_t^T \lambda_C(u)D_{r+\lambda_B+\lambda_C}(t,u)(V_\phi(u)-g_C(u))du\right] && (19)\\ FTDDVA &= -\mathbb{E}_t\left[\int_t^T \lambda_B(u)D_{r+\lambda_B+\lambda_C}(t,u)(V_\phi(u)-g_B(u))du\right] && (20)\\ FVVMVA &= -\mathbb{E}_t\left[\int_t^T (r(u)-r_\phi(u))D_{r+\lambda_B+\lambda_C}(t,u)(V_\phi(u)+\phi(u))du\right] && (21)\\ FVIMVA &= -\mathbb{E}_t\left[\int_t^T (r(u)-r_\psi(u))D_{r+\lambda_B+\lambda_C}(t,u)\psi_B(u)du\right]. && (22) \end{aligned}$$

The sum of $FTDCVA$ and $FTDDVA$ are often referred to as the *bilateral credit value adjustment*, or $BCVA$. In section 6 we provide interpretations for these valuation adjustments, and in particular we explain the origins of the $FVVMVA$ and $FVIMVA$ metrics as these may be less familiar.

6 Model interpretation

In this section we interpret the shareholder and firm valuation adjustments. To make our points we follow Green [10] and set

$$\begin{aligned} g_B(t) &= -(\phi(t)+\psi_B(t)) + (V_\phi(t)+\phi(t)+\psi_B(t))^+ + R_B(V_\phi(t)+\phi(t)+\psi_B(t))^- \\ g_C(t) &= -(\phi(t)+\psi_C(t)) + R_C(V_\phi(t)+\phi(t)+\psi_C(t))^+ + (V_\phi(t)+\phi(t)+\psi_C(t))^- \end{aligned} \tag{23}$$

so there is no margin period of risk or any other of the complexities discussed in Andersen, Phytkin and Sokol [3]. Simplifying further we assume that the bank has entered into a (partially) collateralised trade with the counterparty and is hedging it back-to-back with a clearing house on a fully collateralised basis with collateral rate r. Consequently V_r is both the mark-to-market of the hedge trade and the amount of collateral posted by the bank with the clearing house. From an organisational point of view it makes sense for the bank to let the originating trading desk (e.g. a swaps desk) manage V_r while a special valuation adjustment desk manages the different valuation adjustments.

6.1 The collateral valuation adjustment

If the boundary conditions (23) hold and the bank and counterparty have agreed a gold standard variation margin only CSA such that (a) $\phi(t) = -V_\phi(t)$ and (b) $\psi_B(t) = \psi_C(t) = 0$, then it is straightforward to show that all the shareholder and firm valuation adjustments vanish and $\hat{V} = \hat{V}_{\mathrm{FV}} = V_\phi$. Interestingly, if the boundary condition g_B is different, e.g. since netting does not apply at bank default, but the other conditions continue to hold, then the shareholder value still satisfies $\hat{V} = V_\phi$ whilst the firm value may no longer equal V_ϕ.

If $\hat{V} = V_\phi$ we can interpret $COLVA$ as the difference in value between the total value and its back-to-back hedge caused by the difference in collateralisation. In this case the bank posts or receives the collateral V_r at the rate r to or from the hedge counterparty, and receives or posts V_ϕ at the rate r_ϕ from or to the counterparty. It seems the bank would have to resort to unsecured funding to plug the gap, but there is a way around this dilemma. First the bank and hedge counterparty enter into a r vs r_ϕ basis swap with variable notional V_ϕ. This swap is itself fully collateralised at the rate r so it follows that its value is given by the integral formula (15) and thus equals $COLVA$. The collateral balances and interest payments (from collateral and the basis swap) are summarised in Table 2 and it is clear that the bank is flat so the basis swap removes any need for unsecured funding. Put differently, the $COLVA$ is the value of a basis swap that aligns the hedge counterparty and counterparty CSA terms as illustrated in Figure 1.

	Hedge Counterparty	Basis swap	Counterparty
Balance	V_r	$COLVA$	$-V_\phi$
Interest	$rV_r dt$	$(r_\phi - r)V_\phi dt + rCOLVAdt$	$-r_\phi V_\phi dt$
		$= r_\phi V_\phi dt - rV_r dt$	

Table 2: Bank collateral balances and cashflows for the fully collateralised case. The collateral balance $COLVA$ for the basis swap is either on the balance sheet of the bank or the hedge counterparty depending on the sign but displayed separately in the basis swap column for clarity.

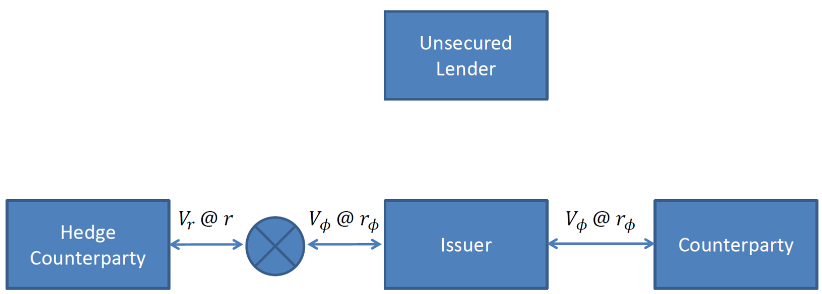

Figure 1: Casflows in the fully collateralised case. The basis swap is denoted by a $\otimes$

6.2 Shareholder value metrics

Here we discuss the origin of the $FCVA$, FVA and MVA and for simplicity we first assume that the counterparty is default free such that $\lambda_C = 0$. In the partially collateralised case the bank needs to fill the gap $V_\phi + \phi + \psi_B$ with unsecured funding as shown in Figure 2 which gives rise to FVA and MVA. Analogously to $COLVA$ these can be interpreted as r_F vs r_ϕ and r_F vs r_ψ basis swaps on the notionals (i.e. unsecured funding requirements) $V_\phi + \phi$ and ψ_B, respectively. These swaps are internal between the trading and the valuation adjustment desks which can be seen that the discounting in the integral formulas is done at the rate r_F.

The addition of counterparty risk has two effects. First there is an $FCVA$ that comes from the cost of hedging the exposure $\hat{V} - g_C = U + (1 - R_C)(V_\phi + \phi + \psi_C)^+$. Second, the fact that the bank has to hedge the loss of the remaining valuation adjustment U implies that the FVA and MVA integrals both contain something that looks like a counterparty survival probability. Moreover,

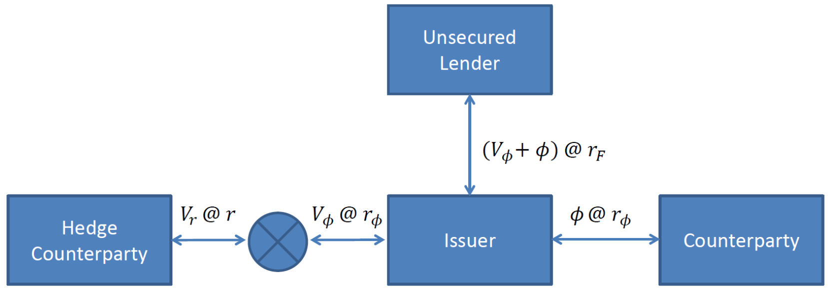

Figure 2: Casflows in the partially collateralised case. The basis swap is denoted by a $\otimes$

the hedging of the adjustments themselves must also be funded and this is why the shareholder valuation adjustments, including $FCVA$ are all discounted with the rate r_F (as opposed to $r + \lambda_B$ for the firm value adjustments). Finally we note that $FCVA$ and MVA are always liabilities, whereas FVA can be either an asset or a liability depending on whether the portfolio and its variation margin collateral generates or requires cash.

6.3 Firm value metrics

As discussed the firm value is the combined value to the shareholders and bondholders of the derivative portfolio, or equivalently the the price someone would pay to take over the derivative portfolio plus the debt funding it. The first-to-default CVA and DVA are well known and represent the reduction and increase in firm value of not being paid and not having to pay respectively. Note that the latter is a benefit to the bondholders of the bank, who unlike the shareholders are still around after the default. As a consequence the boundary condition g_B is present in the firm value but not he shareholder value.

The $FVVMVA$ and $FVIMVA$ are firm value analogues of FVA and MVA with the difference that the funding rate is replaced by the risk-free rate r. Positive collateral spreads $r_\phi - r$ benefits the party that posts variation margin collateral at the expense of the party receiving. Interest on initial margin collateral on the other hand is paid by a third party custodian so increasing the rate r_ψ can increase the firm value of both the bank and the counterparty whilst maintaining the firm value symmetry.

7 Model usage

The model we have developed so far is strictly speaking only valid under the following assumptions laid out in Sections 3 and 4:

1. Frictionless continuous time and amount trading.
2. All trade cash flows, collateral and hedge assets are denominated in a single (domestic) currency.

3. All hedge assets are traded on a collateralised (or repo) basis.
4. The interest rate and spot asset follow a hybrid one-factor IR and local volatility dynamics.
5. The market risk factors are independent of J_B and J_C (e.g. no jump in S or r at counterparty default).
6. Deterministic funding, Libor, collateral and credit spreads.
7. Single bond funding strategy used.
8. Full re-hypothecation of variation margin collateral.
9. Initial margin collateral is held by a third party custodian who pays the interest on it.
10. No basis spreads between bank debt of different seniorities.
11. Deterministic recovery rates for bank and counterparty debt and derivatives (these recovery rates can be different).

Generalising the replication methodology to an arbitrary number of assets with more general dynamics (e.g. stochastic volatility or more interest rate factors) and trades with arbitrary trade events is straightforward albeit laborious. It is however not difficult to see that the valuation adjustment formulas should hold in more general situations. For example, adding more asset classes (denominated in the same currency) and more general dynamics merely expands the operator $\mathcal{A}$ but keeps the adjustment formulas intact. Thus it should be possible to specify the dynamics of the risk-factors directly under the chosen pricing measure and compute the required profiles using Monte-Carlo. On the other hand, stochastic credit or funding spreads are not supported and nor are multiple currencies.

The additivity property of the shareholder value across netting sets discussed in Burgard and Kjaer [9] continues to hold as it is a property of the single bond funding strategy and does not depend on the operator $\mathcal{A}$. This implies that the calculations presented in this paper can be applied per counterparty and the results aggregated to book level.

8 Conclusion

We used a semi-replication approach to derive consistent XVA formulas for shareholder and firm values in a single currency economy with counterparty risk, collateral and funding costs. By consistent we mean that the sum of the valuation adjustment and the reference value equals the total shareholder or firm value, with no missing terms or double counting. Moreover, this approach also ensures that the firm and shareholder values are consistent. The valuation adjustment formulas are derived for the single bond funding strategy of Burgard and Kjaer [8], but since the partial differential equation satisfied by the shareholder value holds for any funding strategy it is possible to derive shareholder value adjustment for other funding strategies.

As in many models used in practise to compute valuation adjustments the risk-free rate, interpreted as the repo-rate on a default-risk free zero coupon bond, is stochastic but the credit, funding, Libor and collateral spreads were deterministic. This added level of complexity does not alter any of the fundamental results derived in Burgard and Kjaer [8], [9], but obviously makes any practical implementation more laborious. To make the model useful in practise it needs to be extended to a multi-currency economy and this is done in Kjaer [12]. Furthermore, the computation rules for the collateral balances ϕ, ψ_B and ψ_C as well as the boundary conditions g_B and g_C also need to be specified.

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A Details of the semi-replication

As described in Section 4.1 we consider the balance sheet of the bank consisting of the derivative portfolio with economic value $\hat{V} = \hat{V}(t, r, S, J_B, J_C)$ and a hedge and funding portfolio Π given by

$$\Pi = \delta_S S + \delta_Z Z_r^T + \delta_C P_C + \sum_j \delta_{F,j} P_{F,j} + \alpha_S \beta_S + \alpha_C \beta_C + \alpha_Z \beta_Z + \alpha_\phi \beta_\phi + \alpha_{\psi,B} \beta_\psi. \tag{24}$$

Our aim is to choose the portfolio weights in (24) in a self-financing way such that $\hat{V} + \Pi = 0$ in all scenarios but possibly bank default. By Section 3 the assets S, P_C, Z_r^T are financed via individual repo-accounts $\beta_S, \beta_C, \beta_Z$ which yields that the *repo-constraints* (7) must hold for all times t strictly before the counterparty default time. Inserting these repo-constraints into the relation $\hat{V} + \Pi = 0$ gives the *funding constraint*

$$\hat{V} + \sum_j \delta_{F,j} P_{F,j} + \alpha_\phi \beta_\phi + \alpha_{\psi,B} \beta_\psi = 0 \tag{25}$$

which must hold at all times strictly before the first of the bank and counterparty default times. Self financing implies that

$$d\Pi = \delta_S dS + \delta_Z dZ_r^T + \delta_C dP_C + \sum_j \delta_{F,j} dP_{F,j} + \alpha_S d\beta_S + \alpha_C d\beta_C + \alpha_Z d\beta_Z + \alpha_\phi d\beta_\phi + \alpha_{\psi,B} d\beta_\psi \tag{26}$$

where we recall that the integrands are evaluated at t^- (i.e. before any jump in J_B or J_C). Next we combine the stochastic differential equations (1) and (5) with the repo-constraints (7) to obtain the *financed asset dynamics*

$$\begin{aligned} \delta_S dS + \alpha_S d\beta_S &= \delta_S (dS - (\gamma_S - q_S) S dt) \\ \delta_Z dZ_r^T + \alpha_Z d\beta_Z &= \delta_Z \frac{\partial Z_r^T}{\partial r} (dr - a_r(t, r) dt) \\ \delta_C dP_C + \alpha_C d\beta_C &= \delta_C (1 - R_C) P_C^- (\lambda_C dt - dJ_C) \end{aligned} \tag{27}$$

where we recall that $\lambda_C = \frac{r_C - \gamma_C}{1 - R_C}$. The right hand side of the last line of (27) can be written as $(1 - R_C) P_C^- (\lambda_C dt - dJ_C) := (s_{CDS} dt - (1 - R_C) dJ_C)$ which can be viewed as an idealised credit default swap where the protection premium rate $s_{CDS} := (1 - R_C) \lambda_C$ per unit of notional is paid continuously. This idealised credit default swap could have been used in the replication in lieu of the risky bond P_C and repo account β_C.

Inserting the right hand sides of (27) along with the dynamics (1) into the hedge portfolio dynamics (26) yields

$$d\Pi = \Big\{ - \delta_Z a_r(t,r)\frac{\partial Z_r^T}{\partial r} + \sum_i \delta_{F,j} P_{F,j}^- r_{F,j} + \delta_C(1-R_C)P_C^-\lambda_C - \delta_S S(\gamma_S - q_S)$$
$$+r_\phi\phi + r_\psi\psi_B)\Big\}dt + \delta_S dS + \delta_Z\frac{\partial Z_r^T}{\partial r}dr - \sum_j \delta_{F,j}(1-R_{F,j})P_{F,j}^- dJ_B - \delta_C(1-R_C)P_C^- dJ_C.$$

By Ito's lemma for general semi-martingales (see e.g. Protter [15]) and the dynamics (1) the economic value $\hat{V} = \hat{V}(t,r,S,J_B,J_C)$ evolves as

$$d\hat{V} = \left\{\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_2\hat{V}\right\}dt + \frac{\partial \hat{V}}{\partial r}dr + \frac{\partial \hat{V}}{\partial S}dS + \Delta\hat{V}_B dJ_B + \Delta\hat{V}_C dJ_C \tag{28}$$

with $\Delta\hat{V}_B := g_B(t) - \hat{V}(t,r,S,0,0)$, $\Delta\hat{V}_C := g_C(t) - \hat{V}(t,r,S,0,0)$ and

$$\mathcal{A}_2 := \frac{1}{2}\sigma_S^2(t,S)S^2\frac{\partial^2}{\partial S^2} + \frac{1}{2}\sigma_r^2(t,r)\frac{\partial^2}{\partial r^2} + \rho\sigma_r(t,r)\sigma_S(t,S)S\frac{\partial^2}{\partial S\partial r}.$$

If we now compare the expressions for $d\hat{V}$ and $d\Pi$ and eliminate the diffusive S and r risks as well as the counterparty default risk by setting $\delta_S = -\frac{\partial \hat{V}}{\partial S}$, $\delta_Z = -\frac{\partial \hat{V}}{\partial r}/\frac{\partial Z_r^T}{\partial r}$ and $\delta_C(1-R_C)P_C^- = \Delta\hat{V}_C$ then the combination of the derivative book and its hedge evolves as

$$d(\hat{V}+\Pi) = \Big\{\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_2\hat{V} + a_r(t,r)\frac{\partial \hat{V}}{\partial r} + (\gamma_S - q_S)S\frac{\partial \hat{V}}{\partial S}$$
$$+\sum_j r_{F,j}\delta_{F,j}P_{F,j}^- + r_\phi\phi + r_\psi\psi_B + \lambda_C(g_C - \hat{V})\Big\}dt$$
$$+\left(g_B - \hat{V} - \sum_j (1-R_{F,j})\delta_{F,j}P_{F,j}^-\right)dJ_B$$

as long as the bank and the counterparty are both alive. If we furthermore define the operator $\mathcal{A} := \mathcal{A}_2 + (\gamma_S - q_S)S\frac{\partial}{\partial S} + a_r(t,r)\frac{\partial}{\partial r}$ and let $\epsilon_h := g_B - \hat{V} - \sum_j (1-R_{F,j})\delta_{F,j}P_{F,j}^-$ then

$$d(\hat{V}+\Pi) = \Big\{\frac{\partial \hat{V}}{\partial t} + \mathcal{A}\hat{V} + \sum_j r_{F,j}\delta_{F,j}P_{F,j}^- + r_\phi\phi + r_\psi\psi_B + \lambda_C(g_C - \hat{V})\Big\}dt + \epsilon_h dJ_B. \tag{29}$$

The no basis condition (3), funding constraint (25) and the definition of ϵ_h imply that

$$\sum_j r_{F,j}\delta_{F,j}P_{F,j} = -r(\hat{V} + \phi + \psi_B) + \lambda_B(g_B - \hat{V}) - \lambda_B\epsilon_h$$

which when inserted into (29) yields the final expression for the (partially) hedged balance sheet dynamics as

$$
\begin{aligned}
d(\hat{V}+\Pi) = & \Big\{\frac{\partial \hat{V}}{\partial t} + \mathcal{A}\hat{V} + \lambda_B(g_B - \hat{V}) + \lambda_C(g_C - \hat{V}) - r(\hat{V}+\phi+\psi_B) + r_\phi\phi + r_\psi\psi_B\Big\}dt \\
& + \epsilon_h(dJ_B - \lambda_B dt).
\end{aligned}
\tag{30}
$$

B Derivation of the valuation adjustments

In this section we derive the shareholder and firm value adjustments. Since they apply strictly before the default of any of the parties we write $P^-_{F,j} = P_{F,j}$ from now on.

B.1 Shareholder value adjustments

We first insert the single bond strategy $\delta_F P_F = -\hat{V} - \phi - \psi_B$ into the balance sheet dynamics (29) before setting the $dt-$terms to zero to obtain

$$
\begin{aligned}
\frac{\partial \hat{V}}{\partial t} + \mathcal{A}\hat{V} - (r_F + \lambda_C)\hat{V} &= -\lambda_C g_C + (r_F - r_\phi)\phi + (r_F - r_\psi)\psi_B \\
\hat{V}(T,r,S,0,0) &= H(r,S).
\end{aligned}
\tag{31}
$$

The partial differential equation (31) is expressed directly in terms of the funding rate r_F rather than λ_B. Next we insert the ansatz $\hat{V} = V_\phi + U$ into (31) and subtract $r_\phi V_\phi$ from both sides to obtain

$$
\begin{aligned}
\frac{\partial V_\phi}{\partial t} + \mathcal{A}V_\phi - (r_F + \lambda_C)V_\phi - r_\phi V_\phi + \frac{\partial U}{\partial t} + \mathcal{A}U - (r_F + \lambda_C)U &= -r_\phi V_\phi - \lambda_C g_C + (r_F - r_\phi)\phi \\
&\quad + (r_F - r_\psi)\psi_B. \\
V_\phi(T,r,S) + U(T,r,S) &= H(r,S).
\end{aligned}
$$

Recognising the PDE (12) with $\xi = r_\phi$ satisfied by V_ϕ on the left hand side allows us to eliminate terms, and after some algebra we get

$$
\begin{aligned}
\frac{\partial U}{\partial t} + \mathcal{A}U - (r_F + \lambda_C)U &= \lambda_C(V_\phi - g_C) + (r_F - r_\phi)(V_\phi + \phi) + (r_F - r_\psi)\psi_B \\
U(T,r,S) &= 0.
\end{aligned}
\tag{32}
$$

Finally the Feynman-Kac theorem gives the solution $U = FCVA + FVA + MVA$ where the valuation adjustments are given in (16) to (18).

The integral formula for $COLVA$ is proved in a similar fashion by inserting the ansatz $V_r = V_\phi - COLVA$ into the PDE (12) with $\xi = r$.

B.2 Firm value adjustments

By Section 4.3 the firm value is given as the solution to the PDE

$$\begin{aligned}\frac{\partial \hat{V}_{\mathrm{FV}}}{\partial t} + \mathcal{A}\hat{V}_{\mathrm{FV}} - (r+\lambda_B+\lambda_C)\hat{V}_{\mathrm{FV}} &= -\lambda_B g_B - \lambda_C g_C - (r_\phi - r)\phi - (r_\psi - r)\psi_B \\ \hat{V}_{\mathrm{FV}}(T,r,S,0,0) &= H(r,S).\end{aligned} \tag{33}$$

The process of deriving the firm value adjustments is very similar to that of the shareholder value adjustments in Section B.1. First the ansatz $\hat{V}_{\mathrm{FV}} = V_\phi + U_{\mathbf{FV}}$ is inserted into (33). Second $r_\phi V_\phi$ is subtracted from both sides of the PDE and the PDE (12) with $\xi = r_\phi$ is used to eliminate terms. This yields

$$\begin{aligned}\frac{\partial U_{\mathrm{FV}}}{\partial t} + \mathcal{A}U_{\mathrm{FV}} - (r+\lambda_B+\lambda_C)U_{\mathrm{FV}} &= \lambda_B(V_\phi - g_B) + \lambda_C(V_\phi - g_C) - (r_\phi - r)(V_\phi + \phi) - (r_\psi - r)\psi_B \\ U_{\mathrm{FV}}(T,r,S) &= 0\end{aligned} \tag{34}$$

so by the Fyenman-Kac theorem $U_{\mathrm{FV}} = FTDCVA + FTDCVA + FVMVVA + FVIMVA$ where the valuation adjustments are given in (19) to (22).

C Multiple netting sets and CSAs

Until now we have assumed that all the trades belong to a single netting set and CSA. We now generalise this and partition the trades of the portfolio into disjoint netting sets NS_l. The trades of a given netting set are then further partitioned into disjoint credit support annexes CSA_k such that trades belonging to the same CSA also belong to the same netting set. The CSA discounted value of the trades per netting set and CSA are denoted $V_{\phi,l}$ and $V_{\phi,k}$ respectively. Furthermore, each CSA has variation and initial margin accounts $\beta_{\phi,k}$ and $\beta_{\psi,k}$ with rates $r_{\phi,k}$ and $r_{\psi,k}$. We can now generalise (24) to

$$\Pi = \delta_S S + \delta_Z Z_r^T + \delta_C P_C + \sum_j \delta_{F,j} P_{F,j} + \alpha_S \beta_S + \alpha_C \beta_C + \alpha_Z \beta_Z + \sum_k \left(\alpha_{\phi,k}\beta_{\phi,k} + \alpha_{\psi,B,k}\beta_{\psi,k}\right). \tag{35}$$

As the closeout process is done per netting set and the proceeds are additive it holds that $g_B = \sum_l g_{B,l}$ and $g_C = \sum_l g_{C,l}$. Applying the semi-replication arguments of Section A to these boundary conditions and the replicating portfolio (35) give the shareholder value as

$$\hat{V} = V_r + \sum_k COLVA_k + \sum_l FCVA_l + \sum_k FVA_k + \sum_k MVA_k$$

with

$$
\begin{aligned}
COLVA_k &= -\mathbb{E}_t\left[\int_t^T (r_{\phi,k}(u) - r(u))D_r(t,u)V_{\phi,k}(u)du\right] \\
FCVA_l &= -\mathbb{E}_t\left[\int_t^T \lambda_C(u)D_{r_F+\lambda_C}(t,u)(V_{\phi,l}(u) - g_{C,l}(u))du\right] \\
FVA_k &= -\mathbb{E}_t\left[\int_t^T (r_F(u) - r_{\phi,k}(u))D_{r_F+\lambda_C}(t,u)(V_{\phi,k}(u) + \phi_k(u))du\right] \\
MVA_k &= -\mathbb{E}_t\left[\int_t^T (r_F(u) - r_{\psi,k}(u))D_{r_F+\lambda_C}(t,u)\psi_{B,k}(u)du\right].
\end{aligned}
$$

Similarly the firm value is given by

$$
\hat{V}_{\mathrm{FV}} = V_r + \sum_k COLVA_k + \sum_l FTDCVA_l + \sum_l FTDDVA_l + \sum_k FVVMVA_k + \sum_k FVIMVA_k
$$

with $COLVA_k$ given above and

$$
\begin{aligned}
FTDCVA_l &= -\mathbb{E}_t\left[\int_t^T \lambda_C(u)D_{r+\lambda_B+\lambda_C}(t,u)(V_{\phi,l}(u) - g_{C,l}(u))du\right] \\
FTDDVA_l &= -\mathbb{E}_t\left[\int_t^T \lambda_B(u)D_{r+\lambda_B+\lambda_C}(t,u)(V_{\phi,l}(u) - g_{B,l}(u))du\right] \\
FVVMVA_k &= -\mathbb{E}_t\left[\int_t^T (r(u) - r_{\phi,k}(u))D_{r+\lambda_B+\lambda_C}(t,u)(V_{\phi,k}(u) + \phi_k(u))du\right] \\
FVIMVA_k &= -\mathbb{E}_t\left[\int_t^T (r(u) - r_{\psi,k}(u))D_{r+\lambda_B+\lambda_C}(t,u)\psi_{B,k}(u)du\right].
\end{aligned}
$$

In summary we obtain credit value adjustments subscripted by the netting set index l and funding/margin value adjustments indexed by the CSA index k. The counterparty level value is then obtained by adding these numbers together.

D Funding curve calibration

Under the single bond funding strategy assumed from Section 5 the bank pays the rate $r_F(t) = r(t) + s_F(t)$ for (variable) overnight funding with recovery R_F. Now assume that the bank can also enter into a so called term funding agreement whereby they agree to pay the fixed spread $s_{F,k}$ over the period $[0, T_k)$. Since the overnight and term funding spreads are both deterministic it is straightforward to show that the relation $\int_0^{T_k} s_F(t)dt = s_{F,k}T_k$ must hold for each tenor T_k which allows us to find a piecewise constant $s_F(t)$ by bootstrapping. Equivalently we may define the risky discount curve $Z_F^T(t)$ by

$$\begin{aligned} Z_F^T(t) &:= \mathbb{E}_t\left[D_{r_F}(t,T)\right] \\ &= Z_r^T(t) D_{s_F}(t,T) \end{aligned}$$

and then calibrate $Z_F^{T_k}(0)$ such that each term deposit reprices at par via the relation $Z_F^{T_k}(t) = \mathbb{E}_t\left[D_{r+s_{F,k}}(t,T_k)\right]$.