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# Consistent XVA Metrics Part II: Multi-currency

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May 10, 2017

**Abstract**

We present a consistent framework for computing shareholder and firm values of derivative portfolios in the presence of collateral, counterparty risk and funding costs in a multi-currency economy. The results extend the single currency economy results from Kjaer [3] and the major difference is that the effective funding spreads now include cross currency basis spreads. This is a consequence of having to hedge the foreign exchange rate risks that arise from converting funding in one currency into collateral in another. The resulting valuation adjustments have been implemented in the forthcoming Bloomberg MARS XVA product.

**Keywords.** Shareholder and firm values, Valuation adjustments, counterparty risk, collateral, CSA discounting, cross currency basis spreads, Bloomberg MARS XVA.

# Contents

# 1 Executive summary

We provide consistent calculations for shareholder and firm value adjustments ("The XVA metrics") for a derivative portfolio between a bank and a counterparty in the presence of counterparty risk, funding costs and collateral support annexes ("CSA") in a multi-currency economy. This setup extends the single currency calculations presented in Kjaer [3] and the resulting valuation adjustments have been implemented in the forthcoming Bloomberg MARS XVA product.

### The XVA metrics

The multi-currency shareholder and firm values $\hat{V}$ and $\hat{V}_{\text{FV}}$ are decomposed in the same way as the single currency counterparties in Kjaer [3] with

$$\begin{aligned}\hat{V} &= V_{\text{OIS}} + COLVA + FCVA + FVA + MVA \\ \hat{V}_{\text{FV}} &= V_{\text{OIS}} + COLVA + FTDCVA + FTDDVA + FVVMVA + FVIMVA\end{aligned}$$

where

- $V_{\text{OIS}}$ and $V_{\text{CSA}}$: the OIS and CSA discounted risk-free values.
- $COLVA$: COLlateral Valuation Adjustment.
- $FCVA$: Funding Curve discounted Credit Valuation Adjustment.
- $FVA$: Funding Valuation Adjustment.
- $MVA$: Margin Valuation Adjustment
- $FTDCVA$: First-To-Default Credit Valuation Adjustment.
- $FTDDVA$: First-To-Default Debit Valuation Adjustment.
- $FVVMVA$: Firm Value Variation Margin Value Adjustment.
- $FVIMVA$: Firm Value Initial Margin Value Adjustment.

Major differences compared to the single currency valuation adjustments:

- The funding spread include the cross currency basis spread between the funding and collateral currencies.
- The funding currency risk-free rate is used for discounting the firm value metrics.

**Credit, funding and CSA curves**

All XVA metrics need the following market data:

- Counterparty credit curve.
- Domestic and trade cash flow currency discount curves, with CSA currency collateral.
- Bank funding currency OIS discount curve.
- Domestic discount curve, with bank funding currency collateral.
- Variation and initial margin collateral spreads per CSA (if applicable).

Shareholder value metrics only:

- Bank unsecured (i.e. risky) discount curve denominated in a chosen funding currency.

Firm value metrics only:

- Bank credit curve.

**Other assumptions**

1. Frictionless continuous time and amount trading.
2. All hedge assets are traded on a collateralised (or repo) basis.
3. The market risk factors are independent of $J_B$ and $J_C$.
4. Single bond funding strategy with a single funding currency used.
5. Full re-hypothecation of variation margin collateral.
6. Initial margin collateral is held by a third party custodian who pays the interest on it.
7. The variation and initial margin accounts may have multiple (and different) eligible currencies with full substitution rights.
8. No basis spreads between bank debt of different seniorities. Cross currency basis between bank debt in different currencies.
9. Deterministic recovery rates for bank and counterparty debt and derivatives (these recovery rates can be different).

## 2 Introduction

In Kjaer [3] we derive consistent shareholder and firm value XVA metrics for a derivative portfolio between a bank and a counterparty in a single currency economy in the presence of variation and initial margin collateral, counterparty risk and funding costs. This paper extends these results to a multi-currency economy where portfolio trade cash flows, funding and collateral can be denominated in different currencies. Intuitively, if one raises funds in one currency to post as collateral in another the cross currency basis spread between the funding and collateral currencies should be included in the effective funding rate. As seen in Table 1 cross currency basis spreads may be sizable and this is our motivation for modelling them.

| Term | Ticker | Pay | Receive | Mid |
|---|---|---|---|---|
| 3 M | JYBSC | -32 | -26 | -29 |
| 6 M | JYBSF | -33 | -27 | -30 |
| 9 M | JYBSI | -35 | -29 | -32 |
| 1 Yr | JYBS1 | -41 | -31 | -36 |
| 2 Yr | JYBS2 | -49 | -41 | -45 |
| 3 Yr | JYBS3 | -56 | -50 | -53 |
| 4 Yr | JYBS4 | -62 | -56 | -59 |
| 5 Yr | JYBS5 | -65 | -64 | -64 |
| 6 Yr | JYBS6 | -68 | -67 | -67 |
| 7 Yr | JYBS7 | -72 | -66 | -69 |
| 8 Yr | JYBS8 | -72 | -66 | -69 |
| 9 Yr | JYBS9 | -71 | -65 | -68 |
| 10 Yr | JYBS10 | -66 | -65 | -66 |
| 12 Yr | JYBS12 | -64 | -58 | -61 |
| 15 Yr | JYBS15 | -56 | -50 | -53 |
| 20 Yr | JYBS20 | -45 | -42 | -44 |
| 30 Yr | JYBS30 | -34 | -31 | -32 |

Table 1: USDJPY cross currency basis spreads in basis points on 03-Feb-2015. Source: Bloomberg

The methodology and notation of this paper deliberately follow that of Kjaer [3] closely. Many, but not all, of the results and conclusions remain unchanged when moving to a multi-currency economy and will not be repeated. Hence it is important to read and understand Kjaer [3] before turning to this paper. The resulting XVA metrics are implemented in the forthcoming Bloomberg MARS XVA product.

This paper is organised as follows: The assets and accounts of the multi-currency economy are introduced in Section 3. In Section 4 we then derive partial differential equations for the shareholder and firm values for a general multi-currency funding strategy, and generalise the reference value definition to a multi-currency setup. In Section 5 we fix a funding strategy with one bond denominated in the funding currency. We interpret the results in Section 6 before discussing the model assumptions in Section 7. We conclude in Section 8. The detailed derivation of the semi-replication and valuation adjustments are given in Appendix A and B. Multiple netting sets and CSAs are discussed in Appendix C. The invariance with respect to the reporting currency under certain circumstances is proved in Appendix D.

## 3 Hedging assets, accounts and rates

We consider a portfolio of derivative trades between a bank $B$ and a counterparty $C$ in a multi-currency economy with assets, accounts and rates given in Table 2. For simplicity we assume all trades belong to a single netting set and credit support annex. The extension to multiple netting sets and CSAs is straightforward and discussed in Appendix C.

| | |
|---|---|
| $i$ | Currency index $0 \le i < N$ with $0 \le d < N$ being the domestic currency index |
| $X_i$ | Currency $i$ FX spot rate. By definition $X_d = 1$ at all times. |
| $X$ | $X = (X_0, X_1, \ldots, X_{N-1})$ |
| $Z_{r,i}^{T_n}$ | Currency $i$ default risk-free zero-coupon (maturing at $T_n$, $n = 0, 1$) bond price. |
| $\beta_{Z,i}, r_i$ | Currency $i$ repo-account and rate secured against $Z_{r,i}^{T_0}$. |
| $r$ | $r = (r_0, r_1, \ldots, r_{N-1})$ |
| $\beta_{X,i}, \gamma_{X,i}$ | Domestic repo-account and rate secured against $Z_{r,i}^{T_1}$. |
| $r_{L,i}$ | Currency $i$ continuously compounded Libor rate used for e.g. swap fixings. |
| $S_i, q_i$ | Currency $i$ denominated spot asset price with dividend yield $q_i$. |
| $S$ | $S = (S_0, S_1, \ldots, S_{N-1})$ |
| $\beta_{S,i}, \gamma_{S,i}$ | Currency $i$ repo-account and rate secured against $S_i$. |
| $P_C, r_C$ | Currency $i_C$ counterparty overnight bond price and rate. |
| $\beta_C, \gamma_C$ | Currency $i_C$ repo-account and rate secured against $P_C$. |
| $P_{F,i,j}, r_{F,i,j}$ | Currency $i$ bank un-secured overnight bond price and rate for seniority $j$. |
| $\beta_{\phi,i}, r_{\phi,i}$ | Currency $i$ variation margin collateral account and rate. |
| $\beta_{\psi,i}, r_{\psi,i}$ | Currency $i$ initial margin collateral account and rate. |

Table 2: Assets, accounts and rates of the multi-currency economy. The bank bond index $j$ is ordered by increasing seniority per currency. The repo and collateral account values are per unit of account.

All assets are traded on repo (i.e. on a fully collateralised) basis. In order to hedge currency $i$ interest rate and foreign exchange risk we need two zero coupon bonds $Z_{r,i}^{T_n}$, $n = 0, 1$ with different maturities, one of which is collateralised in its own currency with rate $r_i$ and the other which is collateralised in domestic currency with rate $\gamma_{X,i}$. This setup is very similar to that proposed in Piterbarg [4] and from the definition of these rates $\gamma_{X,d} = r_d$ for the domestic currency. For each currency $i$ the bank has multiple overnight bonds $P_{F,i,j}$ with different seniorities (and recovery rates) $R_{F,i,j}$ for funding purposes.

The assets and rates of Table 2 are assumed to follow the real world dynamics

$$\begin{aligned} dS_i(t) &= \mu_{S,i}(t, S_i(t))S_i(t)dt + \sigma_{S,i}(t, S_i(t))S_i(t)dW_{S,i}(t) \\ dX_i(t) &= \mu_{X,i}(t, X_i(t))X_i(t)dt + \sigma_{X,i}(t, X_i(t))X_i(t)dW_{X,i}(t) \\ dr_i(t) &= \mu_{r,i}(t, r_i(t))dt + \sigma_{r,i}(t, r_i(t))dW_{r,i}(t) \\ dP_C(t) &= r_C(t)P_C(t^-)dt - (1 - R_C)P_C(t^-)dJ_C(t) \\ dP_{F,i,j}(t) &= r_{F,i,j}(t)P_{F,i,j}(t^-)dt - (1 - R_{F,i,j})P_{F,i,j}(t^-)dJ_B(t) \end{aligned} \tag{1}$$

where $W_{S,i}$, $W_{X,i}$ and $W_{r,i}$ are Wiener-processes with correlations $\rho_{S_i,S_j}$, $\rho_{X_i,X_j}$, $\rho_{r_i,r_j}$, $\rho_{S_i,X_j}$, $\rho_{S_i,r_j}$ and $\rho_{X_i,r_j}$. The processes $J_B$ and $J_C$ are independent Poisson processes. The recovery rates $0 \leq R_C \leq 1$, $0 \leq R_{F,i,0} < R_{F,i,0} < \ldots \leq 1$ are constant and the functions $\mu_{S,i}(t,s)$, $\sigma_{S,i}(t,s)$, $\mu_{X,i}(t,s)$, $\sigma_{X,i}(t,s)$, $\mu_{r,i}(t,r)$, and $\sigma_{r,i}(t,r)$ satisfy standard technical conditions to guarantee strong solutions of the stochastic differential equations for $S_i$, $X_i$ and $r_i$. For the domestic currency we set $\sigma_{X,d}(t, X_d(t)) = 0$ and $X_d(t) = 1$ for all $t$.

The accounts of Table 2 follow the real world dynamics

$$\begin{aligned} d\beta_{S,i}(t) &= (\gamma_{S,i}(t) - q_{S,i}(t))\beta_{S,i}(t)dt \\ d\beta_{Z,i}(t) &= r_i(t)\beta_{Z,i}(t)dt \\ d\beta_{X,i}(t) &= \gamma_{X,i}(t)\beta_{X,i}(t)dt \\ d\beta_C(t) &= \gamma_C(t)\beta_C(t)dt \\ d\beta_{\phi,i}(t) &= r_{\phi,i}(t)\beta_{\phi,i}(t)dt \\ d\beta_{\psi,i}(t) &= r_{\psi,i}(t)\beta_{\psi,i}(t)dt \end{aligned} \tag{2}$$

where the spot asset dividend yields $q_{S,i}$ are deterministic and all other rates are deterministic spreads over the relevant $r_i$ rate such that

$$\begin{aligned} r_{L,i}(t) &= r_i(t) + s_{L,i}(t) \\ r_C(t) &= r_{i_C}(t) + s_C(t) \\ r_{F,i,j}(t) &= r_i(t) + s_{F,i,j}(t) \\ \gamma_C(t) &= r_{i_C}(t) + s_{\beta,C}(t) \\ \gamma_{S,i}(t) &= r_i(t) + s_{\beta,S_i}(t) \\ \gamma_{X,i}(t) &= r_d(t) + s_{X,i}(t) \\ r_{\phi,i}(t) &= r_i(t) + s_{\phi,i}(t) \\ r_{\psi,i}(t) &= r_i(t) + s_{\psi,i}(t). \end{aligned} \tag{3}$$

The spreads $s_{F,i,j}$ and $s_{X,i}$ are the funding and cross-currency basis spreads respectively and as we will see these will appear together in the valuation adjustment formulas. In the absence of bond basis it is straightforward to show that

$$s_{F,i,j} = (1 - R_{F,i,j})\lambda_B \tag{4}$$

where $\lambda_B$ is the deterministic spread of a (potentially hypothetical) bank zero recovery bond independently of the denomination currency. Analogously the spread of a counterparty zero recovery bond is given by $\lambda_C := \frac{r_C - \gamma_C}{1-R_C}$.

As in the single currency model in Kjaer [3] we follow standard short rate modelling for one-factor models and assume that $Z^{T_n}_{r,i}(t) = Z^{T_n}_{r,i}(t, r_i(t))$ so so Itô's Lemma yields that

$$dZ^{T_n}_{r,i}(t) = \left( \frac{\partial Z^{T_n}_{r,i}}{\partial t}(t, r_i(t)) + \frac{1}{2}\sigma^2_{r,i}(t, r_i(t)) \frac{\partial^2 Z^{T_n}_{r,i}}{\partial r_i^2}(t, r_i(t)) \right) dt + \frac{\partial Z^{T_n}_{r,i}}{\partial r_i}(t, r_i(t)) dr_i(t) \tag{5}$$

which shows that this bond can be used to hedge currency $i$ interest rate risk. Next we introduce the *market price of interest rate risk* $\nu_{r,i}(t, r_i)$ given by

$$\nu_{r,i}(t, r_i) := \frac{\frac{\partial Z^{T_n}_{r,i}}{\partial t} + \frac{1}{2}\sigma^2_{r,i}(t, r_i)\frac{\partial^2 Z^{T_n}_{r,i}}{\partial r_i^2} + \mu_{r,i}(t, r_i)\frac{\partial Z^{T_n}_{r,i}}{\partial r_i} - r_i Z^{T_n}_{r,i}}{\sigma_{r,i}(t, r_i)\frac{\partial Z^{T_n}_{r,i}}{\partial r_i}}$$

and let $a_{r,i}(t, r_i) := \mu_{r,i}(t, r_i) - \nu_{r,i}(t, r_i)\sigma_{r,i}(t, r_i)$. The bond dynamics (5) can now be rewritten as

$$dZ^{T_n}_{r,i}(t) = r_i(t) Z^{T_n}_{r,i}(t) dt + \frac{\partial Z^{T_n}_{r,i}}{\partial r_i}(t, r_i(t)) \left( dr_i(t) - a_{r,i}(t, r_i(t)) dt \right). \tag{6}$$

In practise we would typically specify some a-priori forms of $a_{r,i}(t, r_i)$ and $\sigma_{r,i}(t, r)$ to yield the Vasicek, Hull-White, CIR or other one-factor short rate model, and then calibrate it to e.g. market discount factors, caps and swaptions.

As the instantaneous Libor spreads $s_{L,i}(t)$ are deterministic we can compute currency $i$ forward Libor rates with tenor $\tau$ as $L_i(t, T, T+\tau) = \left( D_{s_{L,i}}(T, T+\tau) \frac{Z^T_{r,i}(t)}{Z^{T+\tau}_{r,i}(t)} - 1 \right) \times \frac{1}{\tau}$. Consequently $L_i(t, T, T+\tau)$ is a function of $r_i(t)$.

For the remainder of this paper we suppress the explicit dependence on $t$ to improve the clarity of the exposition and write $P^-_{F,i,j} := P_{F,i,j}(t^-)$ and $P^-_C := P_C(t^-)$.

**Remark:** We hedge foreign exchange risk by using the foreign bond $Z^{T_1}_{r,i}$ as collateral for a domestic loan with rate $\gamma_{X,i}$, which can be seen as an extension of the classic Garman-Kohlhagen [2] approach. As a consequence the cross currency basis spreads $s_{X,i}$ are applied to the domestic OIS rate $r_d$ such that $\gamma_{X,i} = r_d + s_{X,i}$. We could equally well have used a domestic bond $Z^{T_i}_{r,d}$ as collateral for a currency $i$ loan with rate $\bar{\gamma}_{X,i} = r_i + \bar{s}_{X,i}$ to ensure the cross currency basis spreads are applied to the foreign rates instead. It is possible to prove that these two setups yield the same final numerical results and only differ in terms of cross currency basis quotation convention provided that $\gamma_{X,i} - r_i = r_d - \bar{\gamma}_{X,i}$ or equivalently that $\bar{s}_{X,i} = -s_{X,i}$.

# 4 Valuation by semi-replication

We consider a derivative portfolio whose trades pay the total amount $H(r(T), S(T), X(T))$ in domestic currency on the same date $T$ provided that the bank and the counterparty are both alive. The total value in domestic currency of the portfolio including funding, collateral and counterparty risk is denoted by $\hat{V}$ and is an asset to the bank if positive. Moreover we let $\hat{V} = \hat{V}(t, r, S, X, J_B, J_C)$ and use the generic boundary conditions $\hat{V}(t, r, S, X, 1, 0) = g_B(t)$ and $\hat{V}(t, r, S, X, 0, 1) = g_C(t)$.

## 4.1 Semi-replication

Extending Kjaer [3] we consider the bank balance sheet consisting of a derivative book with value $\hat{V}$ and a hedging and funding portfolio

$$\Pi = \sum_i \Pi_{S,i} + \sum_i \Pi_{Z,i} + \sum_i \Pi_{X,i} + \Pi_C + \sum_i \Pi_{F,i} + \sum_i \Pi_{\phi,i} + \sum_i \Pi_{\psi,i} \tag{7}$$

with

$$\begin{aligned}
\Pi_{S,i} &= X_i(\delta_{S,i} S_i + \alpha_{S,i}\beta_{S,i}) \\
\Pi_{Z,i} &= X_i(\delta_{Z,i} Z_{r,i}^{T_0} + \alpha_{Z,i}\beta_{Z,i}) \\
\Pi_{X,i} &= \delta_{X,i} X_i Z_{r,i}^{T_1} + \alpha_{X,i}\beta_{X,i} \\
\Pi_C &= X_{i_C}(\delta_C P_C + \alpha_C \beta_C) \\
\Pi_{F,i} &= X_i \sum_j \delta_{F,i,j} P_{F,i,j} \\
\Pi_{\phi,i} &= X_i \alpha_{\phi,i}\beta_{\phi,i} \\
\Pi_{\psi,i} &= X_i \alpha_{\psi,B,i}\beta_{\psi,i}.
\end{aligned} \tag{8}$$

A positive weight means the bank is long the asset or account and we define the collateral account balances per currency as $\phi_i := \alpha_{\phi,i}\beta_{\phi,i}$, $\psi_{B,i} = \alpha_{\psi,B,i}\beta_{\psi,i}$ and $\psi_{C,i} = \alpha_{\psi,C,i}\beta_{\psi,i}$ with $\alpha_{\psi,B,i} \geq 0$ and $\alpha_{\psi,C,i} \leq 0$.

As in the single currency setup repo-financing and the requirement that $\hat{V} + \Pi = 0$ except possibly at bank default imply that the funding weights $\delta_{F,i,j}$ must satisfy the multi-currency funding constraint

$$\hat{V} + \sum_i \Pi_{F,i} + \sum_i \Pi_{\phi,i} + \sum_i \Pi_{\psi,i} = 0 \tag{9}$$

We can now use Itô's lemma, the boundary conditions $g_B$ and $g_C$, the funding constraint (9), the no-basis condition (4) and choose the hedge ratios $\delta_{S,i}$, $\delta_{Z,i}$, $\delta_{X,i}$ and $\delta_C$ such that the market and counterparty default risks are hedged out to obtain the balance sheet dynamics as

$$d(\hat{V}+\Pi) = \Big\{\frac{\partial \hat{V}}{\partial t} + \mathcal{A}\hat{V} + \lambda_B(g_B-\hat{V}) + \lambda_C(g_C-\hat{V}) + \sum_{i,j}\gamma_{X,i}\delta_{F,i,j}X_iP^-_{F,i,j} + \sum_i \gamma_{\phi,i}X_i\phi_i + \sum_i \gamma_{\psi,i}X_i\psi_{B,i}\Big\}dt + \epsilon_h(dJ_B - \lambda_B dt) \tag{10}$$

with the parabolic operator $\mathcal{A}$ given in its full grandeur in Appendix A, effective domestic collateral rates $\gamma_{\phi,i} := r_{\phi,i} - r_i + \gamma_{X,i}$ and $\gamma_{\psi,i} := r_{\psi,i} - r_i + \gamma_{X,i}$ and hedge error at own default $\epsilon_h := g_B - \hat{V} - \sum_{i,j}(1-R_{F,i,j})X_i\delta_{F,i,j}P^-_{F,i,j}$. The full proof is given in Appendix A.

To make the balance sheet dynamics (10) look more like its singe-currency equivalent in Kjaer [3] we first express the funding strategy in terms of the *funding weights* $\omega^F_{ij}$ defined via the relation $\delta_{F,i,j}X_iP_{F,i,j} = -\omega^F_{ij}(\hat{V}+\Pi_\phi+\Pi_\psi)$. The funding constraint implies $\sum_{ij}\omega^F_{ij} = 1$ and w.l.o.g. we set one of the weights to one the others to zero if $\hat{V}+\Pi_\phi+\Pi_\psi = 0$. Similarly we define collateral weights $\omega^\phi_i$ and $\omega^\psi_i$ by $X_i\phi_i =: \omega^\phi_i\phi^d$ and $X_i\psi_{B,i} =: \omega^\psi_i\psi^d_B$ where $\phi^d := \sum_i X_i\phi_i$ and $\psi^d_B := \sum_i X_i\psi_{B,i}$ and the two sets of weights sum to one. With this new notation in place we can now rewrite (10) as

$$d(\hat{V}+\Pi) = \Big\{\frac{\partial \hat{V}}{\partial t} + \mathcal{A}\hat{V} + \lambda_B(g_B-\hat{V}) + \lambda_C(g_C-\hat{V}) - \gamma_{X,E}(\hat{V}+\phi^d+\psi^d_B) + \gamma_{\phi,E}\phi^d + \gamma_{\psi,E}\psi^d_B\Big\}dt + \epsilon_h(dJ_B - \lambda_B dt) \tag{11}$$

with *effective domestic risk-free rate* $\gamma_{X,E} := r_d + \sum_{i,j}\omega^F_{ij}s_{X,i}$ and effective collateral rates $\gamma_{\phi,E} := \sum_i \omega^\phi_i\gamma_{\phi,i}$ and $\gamma_{\psi,E} := \sum_i \omega^\psi_i\gamma_{\psi,i}$, respectively. The effective risk-free rate depends on the proportion of the total funding requirement raised in each currency and the magnitude of the cross currency basis spreads. The single currency model by contrast only has one risk-free rate which consequently does not depend on the funding strategy. The effective collateral rates depend on the terms of the credit support annex in place between the bank and the counterparty.

### 4.2 Shareholder and firm values

The definitions of shareholder and firm values discussed at length in Kjaer [3] remain valid after the extension to multiple currencies. Hence the partial differential equation satisfied by the shareholder value $\hat{V}$ is obtained by setting all the $dt-$terms in the balance sheet dynamics (11) to zero to ensure it is self-financing to the shareholders while the bank is alive. We thus have that

$$\begin{aligned}\frac{\partial \hat{V}}{\partial t} + \mathcal{A}\hat{V} - (\gamma_{X,E}+\lambda_B+\lambda_C)\hat{V} &= -\lambda_B g_B - \lambda_C g_C - (\gamma_{\phi,E}-\gamma_{X,E})\phi^d - (\gamma_{\psi,E}-\gamma_{X,E})\psi^d_B + \lambda_B\epsilon_h \\ \hat{V}(T,r,S,X,0,0) &= H(r,S,X)\end{aligned} \tag{12}$$

which is very similar in form to the single currency equivalent in Kjaer [3]. Note that depending on the funding strategy $\gamma_{X,E}$ may depend on $t, r, S, X$ or even $\hat{V}$ itself so the PDE (12) may be implicit.

The partial differential equation satisfied by the firm value $\hat{V}_{\text{FV}}$ is obtained by removing the $\lambda_B \epsilon_h$ term on the right hand side of (12). For the firm value to be symmetric between the bank and the counterparty it must hold that (a) $\psi_{B,i} = -\psi_{C,i}$ for each currency $i$ like in the single currency economy, and (b) that both parties use the same currency mix in their respective funding strategies such that their effective risk-free rates are equal.

### 4.3 Risk neutral dynamics

By the Feynman-Kac theorem the solution to (12) can be expressed in terms of an expectation $\mathbb{E}_t\left[\right]$ with respect to a domestic probability measure $\mathbb{Q}$ under which $J_B(t)$ and $J_C(t)$ are independent Poisson processes with intensities $\lambda_B(t)$ and $\lambda_C(t)$ and

$$
\begin{aligned}
dS_i(t) &= \{\gamma_{S,i}(t) - q_{S,i}(t) - \rho_{S_i,X_i}\sigma_{S,i}(t,S_i(t))\sigma_{X,i}(t,X_i(t))\}S_i(t)dt + \sigma_{S,i}(t,S_i(t))S_i(t)dW^{\mathbb{Q}}_{S,i}(t) \\
dX_i(t) &= \{\gamma_{X,i}(t) - r_i(t)\}X_i(t)dt + \sigma_{X,i}(t,X_i(t))X_i(t)dW^{\mathbb{Q}}_{X,i}(t) \\
dr_i(t) &= \{a_{r,i}(t,r_i(t)) - \rho_{X_i,r_i}\sigma_{r,i}(t,r_i(t))\sigma_{X,i}(t,X_i(t))\}dt + \sigma_{r,i}(t,r_i(t))dW^{\mathbb{Q}}_{r,i}(t)
\end{aligned} \tag{13}
$$

where $W^{\mathbb{Q}}_{S,i}$, $W^{\mathbb{Q}}_{X,i}$ and $W^{\mathbb{Q}}_{r,i}$ are correlated $\mathbb{Q}$-Wiener processes. Compared to the single currency dynamics in Kjaer [3] the drifts of foreign interest rates $r_i$ and spot assets $S_i$ contain quanto adjustments into the domestic currency $i = d$. The FX spot rate $X_i$ grows at the domestic rate $\gamma_{X,i}$ rather than $r_d$ which is due to the cross currency basis.

### 4.4 Reference values

The multi-currency reference value definition is somewhat complicated by the number of different discounting curves available. To get started we let $\xi$ be an arbitrary domestic rate such that $s_\xi(t) := \xi(t) - r_d(t)$ is deterministic. We then define the rate $\xi$-discounted domestic reference value $V^d_\xi$ to be the Feynman-Kac solution $V^d_\xi(t,r,S,X) = \mathbb{E}_t\left[D_\xi(t,T)H(r(T),S(T),X(T))\right]$ to the partial differential equation

$$
\begin{aligned}
\frac{\partial V^d_\xi}{\partial t} + \mathcal{A}V^d_\xi &= \xi V^d_\xi \\
V^d_\xi(T,r,S,X) &= H(r,S,X).
\end{aligned} \tag{14}
$$

The rationale for writing down the pricing Equation (14) for the domestic value $V^d_\xi$ regardless of the payoff $H$ is that it can be expressed in terms of the operator $\mathcal{A}$ and the rates of table 2. The OIS discounted reference values $V^d_{r,i_r}$ is obtained by setting $\xi(t) = \gamma_{X,i_r}$, where $i_r$ is some reference collateral currency (e.g. the one used by a particular clearing house).

The CSA discounted value $V^d_\phi$ is obtained by setting $\xi = \gamma_{\phi,E}$. For the remainder of this paper we allow multiple eligible currencies with full substitution rights. In this case $\gamma_{\phi,E} = \max_i \gamma_{\phi,i}$ where the index $i$ runs over the set of eligible currencies. It follows that $s_\xi(t) = \max_i(s_{\phi,i}(t) + s_{X,i}(t))$ which is deterministic and allows us to construct a cheapest-to-deliver discounting curve $\gamma_{\phi,E}$ can be constructed up-front from its constituents.

# 5 Funding strategies and valuation adjustments

All but the very largest institutions would raise funding in a single currency so we adapt the single bond strategy from Kjaer [3] by assuming the existence of a exactly one currency $i_F$ bond denoted by $P_{F,i_F}$ (we drop the seniority index $j$). Under these assumptions on the funding strategy and the collateral accounts the funding constraint (9) takes the form

$$X_{i_F}\delta_{F,i_F}P_{F,i_F} = -\hat{V} - \phi^d - \psi^d_B. \tag{15}$$

Our aim is to calculate the valuation adjustments over the OIS discounted reference value $V^d_r := V^d_{r,i_r}(t,r,S,X)$ defined in Section 4.4. In order to preserve space we sometimes suppress the arguments $t,r,S,X$ from values and valuation adjustments. As in the single currency model we write $\hat{V} = V^d_r + COLVA + U$ and $\hat{V}_{\mathrm{FV}} = V^d_r + COLVA + U_{\mathrm{FV}}$ where the *Collateral Valuation Adjustment* $COLVA = V^d_\phi - V^d_r$ can be written on integral form as

$$COLVA = -\mathbb{E}_t\left[\int_t^T (\gamma_{\phi,E}(u) - \gamma_{X,i_r}(u))D_{\gamma_{X,i_r}}(t,u)V^d_\phi(u)du\right] \tag{16}$$

where the expectation is taken with respect to the measure $\mathbb{Q}$ introduced in Section 4.3. The spread inside the integral contains the cross currency basis between the currencies prevailing cheapest to deliver currency and $i_r$. For counterparties without CSA we set $V^d_\phi := V^d_r$ so $COLVA = 0$ in this case.

## 5.1 Shareholder value adjustment metrics

In Appendix B we prove that $U = FCVA + FVA + MVA$ with the *funding curve discounted credit, funding and margin* value adjustments given by

$$\begin{aligned}
FCVA &= -\mathbb{E}_t\left[\int_t^T \lambda_C(u)D_{\gamma_F+\lambda_C}(t,u)(V^d_\phi(u) - g_C(u))du\right] &(17)\\
FVA &= -\mathbb{E}_t\left[\int_t^T (\gamma_F(u) - \gamma_{\phi,E}(u))D_{\gamma_F+\lambda_C}(t,u)(V^d_\phi(u) + \phi^d(u))du\right] &(18)\\
MVA &= -\mathbb{E}_t\left[\int_t^T (\gamma_F(u) - \gamma_{\psi,E}(u))D_{\gamma_F+\lambda_C}(t,u)\psi^d_B(u)du\right] &(19)
\end{aligned}$$

where $\gamma_F := r_{F,i_F} - r_{i_F} + \gamma_{X,i_F}$ is the effective domestic funding rate for currency $i_F$. The valuation adjustments (17) to (19) look very much like their single currency equivalents in Kjaer [3] except that they use effective domestic funding and collateral rates which include cross currency basis spreads. In Section 6 we provide interpretations for these valuation adjustments.

### 5.2 Firm value adjustment metrics

In Appendix B we prove that $U_{\mathrm{FV}} = FTDCVA + FTDDVA + FVVMVA + FVIMVA$ with the *first-to-default credit, first-to-default debit, firm value variation margin and firm value initial margin* value adjustments given by

$$
\begin{aligned}
FTDCVA &= -\mathbb{E}_t\left[\int_t^T \lambda_C(u) D_{\gamma_{X,i_F}+\lambda_B+\lambda_C}(t,u)(V_\phi^d(u) - g_C(u))du\right] && (20)\\
FTDDVA &= -\mathbb{E}_t\left[\int_t^T \lambda_B(u) D_{\gamma_{X,i_F}+\lambda_B+\lambda_C}(t,u)(V_\phi^d(u) - g_B(u))du\right] && (21)\\
FVVMVA &= -\mathbb{E}_t\left[\int_t^T (\gamma_{X,i_F}(u) - \gamma_{\phi,E}(u)) D_{\gamma_{X,i_F}+\lambda_B+\lambda_C}(t,u)(V_\phi^d(u) + \phi^d(u))du\right] && (22)\\
FVIMVA &= -\mathbb{E}_t\left[\int_t^T (\gamma_{X,i_F}(u) - \gamma_{\psi,E}(u)) D_{\gamma_{X,i_F}+\lambda_B+\lambda_C}(t,u)\psi_B^d(u)du\right]. && (23)
\end{aligned}
$$

Again the valuation adjustments (20) to (23) look like their single currency equivalents in Kjaer [3] except that the effective domestic risk-free rate $\gamma_{X,E} = \gamma_{X,i_F}$ is used. We discuss any differences further in Section 6.

## 6 Model interpretation

Many of the interpretations of the single currency results in Kjaer [3] remain valid for the multi-currency model. There are also some new features that are unique to the multi-currency setup and they are the focus of this section.

### 6.1 COLVA

The collateral valuation adjustment is still the difference in discounting between the OIS and CSA discounted value and it can be implemented as a basis swap between the bank and the hedge counterparty as described in Section 6 in Kjaer [3]. For $COLVA$ to be non-zero in the single currency setup the parties would have agreed on a collateral spread $s_\phi$ in the credit support annex. This changes in the multi-currency setup where $COLVA$ becomes a cross currency basis swap between currency $i_r$ and the cheapest to deliver collateral currency. It is necessary to set up this basis swap to transform the cash flows from unsecured funding and counterparty collateral into the currency of the hedge counterparty collateral.

### 6.2 Shareholder value metrics

Similarly the $FVA$ now uses the effective funding spread $\gamma_F - \gamma_{\phi,E} = s_{F,i_F} + s_{X,i_F} - \max_i(s_{\phi,i_\phi} + s_{X,i_\phi})$ which remains deterministic but includes the cross currency basis between the funding currency $i_F$ and the cheapest to deliver collateral currency to reflect the cost or gain of transforming funds between them. The same applies to the spread $\gamma_F - \gamma_{\psi,E}$ used for the $MVA$. It follows that it is possible to eliminate the basis spread dependence from the funding spreads if the funding and collateral currencies are identical. All the shareholder value adjustments are discounted with $\gamma_F + \lambda_C$ as the hedging of the adjustments themselves use the same funding strategy as the risk-free portfolio value $V_\phi^d$.

### 6.3 Firm value metrics

The firm value metrics also look very similar to their single currency counterparts in Kjaer [3]. One major difference is that the effective domestic risk-free rate $\gamma_{X,i_F}$ is used for discounting rather than $r_d$. The dependence on the funding currency basis spread in the discounting implies that the $BCVA$ is only symmetric between the issuer and the counterparty if they use the same funding strategy and currency, or equivalently, use the same risk-free rate. In the multi-curve setup of Section 3 it is not equivalent to raise funds at the risk-free rate in one currency and swap into domestic and raising funds in domestic currency directly.

Another difference is that the spread used for $FVVMVA$ and $FVIMVA$ now include cross currency basis spreads between the funding and collateral currencies.

### 6.4 Reporting currency invariance

We will conclude this section by discussing the dependency of the reporting currency. More specifically assume that we embed a single currency $i$ economy in our multi-currency economy such that (a) $H(r, S, X) = X_i H_i(r_i, S_i)$, (b) $i_\phi = i_\psi = i_F = i_C = i$, (c) $g_B = X_i g_{B,i}$ and (d) $g_C = X_i g_{C,i}$. In other words, all trade, collateral and repo account cash flows are denominated in currency $i$, and closeout at default is determined in currency $i$. Under these assumptions we expect to be able to calculate the reference value, shareholder and firm value metrics in a currency $i$ single currency model and convert the results into domestic currency at the current spot exchange rate $X_i$. In particular we do not expect any of the results to depend on the cross currency basis spread $s_{X,i}$. That this is indeed the case is proved in Appendix D. We conclude by noting that if $FTDCVA$ and $FTDDVA$ had used the domestic risk-free rate $r_d$ rather than $\gamma_{X,i_F}$ for discounting then this result would not have been true and these metrics would have shown a sensitivity to the cross currency basis spread $s_{X,i}$.

## 7 Model usage

The model we have developed so far is strictly speaking only valid under the following assumptions laid out in Sections 3 to 6:

1. Frictionless continuous time and amount trading.
2. All hedge assets are traded on a collateralised (or repo) basis.
3. The interest rates, spot assets and foreign exchange rates follow a multi-currency hybrid one-factor IR and local volatility dynamics.

4. The market risk factors are independent of $J_B$ and $J_C$.
5. Deterministic cross currency basis, funding, Libor-OIS, collateral and credit spreads.
6. Single bond funding strategy with a single funding currency used.
7. Full re-hypothecation of variation margin collateral.
8. Initial margin collateral is held by a default free third party custodian who pays the interest on it.
9. The variation and initial margin accounts may have multiple (and different) eligible currencies with full substitution rights.
10. No basis spreads between bank debt of different seniorities. Cross currency basis between bank debt in different currencies.
11. Deterministic recovery rates for bank and counterparty debt and derivatives (these recovery rates can be different).

As mentioned in Kjaer [3] the model can be extended to more asset classes and more advanced market dynamics without affecting the structure of the valuation adjustments. Moreover, the additivity property of Burgard and Kjaer [1] holds for the chosen funding strategy. Hence the calculations presented in this paper can be applied per counterparty and the results aggregated to book level.

## 8 Conclusion

We extended the results in Kjaer [3] for a single currency economy to include multi-currency trades, funding and collateral. This extension is motivated by the fact that cross-currency basis spreads can be of the same magnitude as bank funding spreads. The resulting valuation adjustment formulas look very much like their single currency counterparts with the major difference that we have to use effective funding and collateral spreads which include cross currency basis spreads. Another difference compared to the single currency setup is that the firm value metrics including the bilateral credit value adjustment should use the funding currency risk-free rate. All the shareholder value and firm value adjustment metrics of Sections 5.1 and 5.2 have been implemented in the forthcoming Bloomberg MARS XVA product.

The model developed in this paper could be used to derive valuation adjustments for more advanced multi-currency funding strategies. Other possible model extensions would be to introduce stochastic credit, funding and cross currency basis spreads as well as introducing a devaluation jump in the counterparty domicile currency upon default.

## References

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[4] V. Piterbarg. Cooking with collateral. *Risk*, 58-63, August 2012.

# A Details of the semi-replication

As described in Section 4 we consider the balance sheet of the bank consisting of the derivative portfolio with economic value $\hat{V} = \hat{V}(t, r, S, X, J_B, J_C)$ and a hedge and funding portfolio $\Pi$ given in Equations (7) and (8). Our aim is to choose the portfolio weights in a self-financing way such that $\hat{V} + \Pi = 0$ in all scenarios but possibly bank default. By Section 3 all the assets except the bank bonds and collateral(s) are financed via individual repo-accounts which implies the *repo-constraints*

$$\begin{aligned} \Pi_{S,i} &= 0 \\ \Pi_{Z,i} &= 0 \\ \Pi_{X,i} &= 0 \\ \Pi_C &= 0 \end{aligned} \tag{24}$$

must hold for all times $t$ strictly before the counterparty default time. We next insert these repo-constraints into the relation $\hat{V} + \Pi = 0$ and obtain the *funding constraint* (9) which must also hold before bank default. Self financing implies

$$\begin{aligned} d\Pi_{S,i} &= \delta_{S,i} d(X_i S_i) + \alpha_{S,i} d(X_i \beta_{S,i}) \\ d\Pi_{Z,i} &= \delta_{Z,i} d(X_i Z_{r,i}^{T_0}) + \alpha_{Z,i} d(X_i \beta_{Z,i}) \\ d\Pi_{X,i} &= \delta_{X,i} d(X_i Z_{r,i}^{T_1}) + \alpha_{X,i} d\beta_{X,i} \\ d\Pi_C &= \delta_C d(X_{i_C} P_C) + \alpha_C d(X_{i_C} \beta_C) \\ d\Pi_{F,i} &= \sum_j \delta_{F,i,j} d(X_i P_{F,i,j}) \\ d\Pi_{\phi,i} &= \alpha_{\phi,i} d(X_i \beta_{\phi,i}) \\ d\Pi_{\psi,i} &= \alpha_{\psi,B,i} d(X_i \beta_{\psi,i}). \end{aligned} \tag{25}$$

We now expand the differentials on the right hand sides of (25) by applying the Itô product rule, then insert the account dynamics (2), and finally invoke the repo constraints (24) to obtain

$$
\begin{aligned}
d\Pi_{S,i} &= \delta_{S,i} X_i (dS_i - (\gamma_{S,i} - q_{S,i}) S_i dt) + \delta_{S,i} d[S_i, X_i] \\
d\Pi_{Z,i} &= \delta_{Z,i} X_i (dZ_{r,i}^{T_0} - r_i Z_{r,i}^{T_0} dt) + \delta_{Z,i} d[Z_{r,i}^{T_0}, X_i] \\
d\Pi_{X,i} &= \delta_{X,i} (Z_{r,i}^{T_1} dX_i + X_i dZ_{r,i}^{T_1} + d[Z_{r,i}^{T_1}, X_i] - \gamma_{X,i} (X_i Z_{r,i}^{T_1}) dt) \\
d\Pi_C &= \delta_C X_{i_C} (dP_C - \gamma_C P_C^- dt) \\
d\Pi_{F,i} &= \sum_j \delta_{F,i,j} (P_{F,i,j}^- dX_i + X_i dP_{F,i,j}) \\
d\Pi_{\phi,i} &= \alpha_{\phi,i} (\beta_{\phi,i} dX_i + r_{\phi,i} X_i \beta_{\phi,i} dt) \\
d\Pi_{\psi,i} &= \alpha_{\psi,B,i} (\beta_{\psi,i} dX_i + r_{\psi,i} X_i \beta_{\psi,i} dt).
\end{aligned} \tag{26}
$$

It is possible to write the third line of (26) as $d\Pi_{X,i} = \delta_{X,i}(d(X_i Z_{r,i}^{T_1}) - \gamma_{X,i}(X_i Z_{r,i}^{T_1})dt)$ which shows that the product $X_i Z_{r,i}^{T_1}$ can be viewed as a domestic asset with repo rate $\gamma_{X,i}$ inline with the classic Garman-Kohlagen model [2] for foreign exchange derivatives. To proceed we use the asset dynamics (1) and (5) to calculate the covariation processes

$$
\begin{aligned}
d[S_i, X_i] &= \rho_{S_i, X_i} \sigma_{X,i} \sigma_{S,i} X_i S_i dt \\
d[Z_{r,i}^{T_n}, X_i] &= \rho_{r_i, X_i} \sigma_{X,i} \sigma_{r,i} X_i \frac{\partial Z_{r,i}^{T_n}}{\partial r_i} dt.
\end{aligned} \tag{27}
$$

We also define the currency $i$ collateral balances $\phi_i := \alpha_{\phi,i}\beta_{\phi,i}$ and $\psi_i^B := \alpha_{\psi,B,i}\beta_{\psi,i}$. Inserting these balances along with the covariations (27), bond dynamics (6) and dynamics (1) for $P_C$ and $P_{F,i,j}$ into (26) then yields

$$
\begin{aligned}
d\Pi_{S,i} &= \delta_{S,i} X_i (dS_i - (\gamma_{S,i} - q_{S,i} - \rho_{S_i,X_i} \sigma_{X,i} \sigma_{S,i}) S_i dt) \\
d\Pi_{Z,i} &= \delta_{Z,i} X_i \frac{\partial Z_{r,i}^{T_0}}{\partial r_i} (dr_i - (a_{r,i} - \rho_{r_i,X_i} \sigma_{X,i} \sigma_{r,i}) dt) \\
d\Pi_{X,i} &= \delta_{X,i} \left( Z_{r,i}^{T_1} dX_i + X_i \frac{\partial Z_{r,i}^{T_1}}{\partial r_i} (dr_i - a_{r,i} dt + \rho_{r_i,X_i} \sigma_{X,i} \sigma_{r,i} dt) - (\gamma_{X,i} - r_i) X_i Z_{r,i}^{T_1} dt \right) \\
d\Pi_C &= \delta_C X_{i_C} P_C^- ((r_C - \gamma_C) dt - (1 - R_C) dJ_C) \\
d\Pi_{F,i} &= \sum_j \delta_{F,i,j} P_{F,i,j}^- (dX_i + X_i (r_{F,i,j} dt - (1 - R_{F,i,j}) dJ_B)) \\
d\Pi_{\phi,i} &= \phi_i (dX_i + r_{\phi,i} X_i dt) \\
d\Pi_{\psi,i} &= \psi_i^B (dX_i + r_{\psi,i} X_i dt).
\end{aligned} \tag{28}
$$

Re-arranging the terms to collect the coefficients in front of each differential $dS_i$, $dr_i$, $dX_i$, $dJ_B$

and $dJ_C$ yields

$$
\begin{aligned}
d\Pi = &\sum_i \delta_{S,i} X_i dS_i + \sum_i X_i \left( \delta_{Z,i} \frac{\partial Z_{r,i}^{T_0}}{\partial r_i} + \delta_{X,i} \frac{\partial Z_{r,i}^{T_1}}{\partial r_i} \right) dr_i \\
&+ \sum_i \left( \delta_{X,i} Z_{r,i}^{T_1} + \sum_j \delta_{F,i,j} P_{F,i,j} + \phi_i + \psi_i^B \right) dX_i \\
&- \sum_j \delta_{F,i,j} (1 - R_{F,i,j}) P_{F,i,j}^- dJ_B - \delta_C X_{i_C} (1 - R_C) P_C^- dJ_C \\
&- \delta_{S,i} X_i S_i (\gamma_{S,i} - q_{S,i} - \rho_{S_i,X_i} \sigma_{X,i} \sigma_{S,i}) dt \\
&- \delta_{Z,i} X_i \frac{\partial Z_{r,i}^{T_0}}{\partial r_i} (a_{r,i} dt - \rho_{r_i,X_i} \sigma_{X,i} \sigma_{r,i}) dt \\
&- \delta_{X,i} X_i \left( \frac{\partial Z_{r,i}^{T_1}}{\partial r_i} (a_{r,i} dt - \rho_{r_i,X_i} \sigma_{X,i} \sigma_{r,i}) - (\gamma_{X,i} - r_i) Z_{r,i}^{T_1} \right) dt \\
&+ \delta_C X_{i_C} P_C^- (r_C - \gamma_C) dt + \sum_j \delta_{F,i,j} P_{F,i,j}^- X_i r_{F,i,j} dt \\
&+ \sum_i (r_{\phi,i} X_i \phi_i + r_{\psi,i} X_i \psi_i^B) dt
\end{aligned} \tag{29}
$$

Ito's Lemma for general semi-martingales combined with the dynamics (1) gives us the dynamics of the derivative portfolio as

$$
d\hat{V} = \left\{ \frac{\partial \hat{V}}{\partial t} + \mathcal{A}_2 \hat{V} \right\} dt + \sum_i \frac{\partial \hat{V}}{\partial r_i} dr_i + \sum_i \frac{\partial \hat{V}}{\partial X_i} dX_i + \sum_i \frac{\partial \hat{V}}{\partial S_i} dS_i + \Delta \hat{V}_B dJ_B + \Delta \hat{V}_C dJ_C \tag{30}
$$

where $\mathcal{A}_2 := \mathcal{A}_{SS} + \mathcal{A}_{XX} + \mathcal{A}_{rr} + \mathcal{A}_{SX} + \mathcal{A}_{Sr} + \mathcal{A}_{Xr}$ with intra-asset class terms

$$
\begin{aligned}
\mathcal{A}_{SS} &:= \frac{1}{2} \sum_{i,j} \rho_{S_i,S_j} \sigma_{S,i}(t, S_i) \sigma_{S,j}(t, S_j) S_i S_j \frac{\partial^2}{\partial S_i \partial S_j} \\
\mathcal{A}_{XX} &:= \frac{1}{2} \sum_{i,j} \rho_{X_i,X_j} \sigma_{X,i}(t, X_i) \sigma_{X,j}(t, X_j) X_i X_j \frac{\partial^2}{\partial X_i \partial X_j} \\
\mathcal{A}_{rr} &:= \frac{1}{2} \sum_{i,j} \rho_{r_i,r_j} \sigma_{r,i}(t, r_i) \sigma_{r,j}(t, r_j) \frac{\partial^2}{\partial r_i \partial r_j}
\end{aligned}
$$

cross asset class terms

$$
\begin{aligned}
\mathcal{A}_{SX} &:= \frac{1}{2}\sum_{i,j}\rho_{S_i,X_j}\sigma_{S,i}(t,S_i)\sigma_{X,j}(t,X_j)S_iX_j\frac{\partial^2}{\partial S_i\partial X_j}\\
\mathcal{A}_{Sr} &:= \frac{1}{2}\sum_{i,j}\rho_{S_i,r_j}\sigma_{S,i}(t,S_i)\sigma_{r,j}(t,r_j)S_i\frac{\partial^2}{\partial S_i\partial r_j}\\
\mathcal{A}_{Xr} &:= \frac{1}{2}\sum_{i,j}\rho_{X_i,r_j}\sigma_{X,i}(t,X_i)\sigma_{r,j}(t,r_j)X_i\frac{\partial^2}{\partial X_i\partial r_j}
\end{aligned}
$$

and $\Delta\hat{V}_B := g_B - \hat{V}$ and $\Delta\hat{V}_C := g_C - \hat{V}$.

By comparing (29) and (30) we see that all the $dS_i$, $dr_i$, $dX_i$ and $dJ_C$ terms in $d(\hat{V}+\Pi)$ can be eliminated if we set

$$
\begin{aligned}
\delta_{S,i} &= -\frac{1}{X_i}\frac{\partial\hat{V}}{\partial S_i}\\
\delta_{X,i} &= -\frac{1}{Z_{r,i}^{T_1}}\left\{\frac{\partial\hat{V}}{\partial X_i} + \sum_j \delta_{F,i,j}P_{F,i,j}^{-} + \phi_i + \psi_{B,i}\right\}\\
\delta_{Z,i} &= -\frac{1}{\frac{\partial Z_{r,i}^{T_0}}{\partial r_i}}\left(\frac{1}{X_i}\frac{\partial\hat{V}}{\partial r_i} + \delta_{X,i}\frac{\partial Z_{r,i}^{T_1}}{\partial r_i}\right)\\
\delta_C &= \frac{g_C - \hat{V}}{(1-R_C)X_{i_C}P_C^{-}}
\end{aligned}
\tag{31}
$$

where we defer expanding $\delta_{X,i}$ inside the expression for $\delta_{Z,i}$ in order to simplify the remaining algebra somewhat. Inserting the hedge ratios (31) into (29) yields

$$
\begin{aligned}
d(\hat{V}+\Pi) = \Big\{&\frac{\partial\hat{V}}{\partial t} + \mathcal{A}_2\hat{V} + \sum_i(\gamma_{S,i} - q_{S,i} - \rho_{X_i,S_i}\sigma_{X,i}\sigma_{S,i})S_i\frac{\partial\hat{V}}{\partial S_i}\\
&+ \sum_i(a_{r,i} - \rho_{X_i,r_i}\sigma_{X,i}\sigma_{r,i})\frac{\partial\hat{V}}{\partial r_i}\\
&+ \sum_i(\gamma_{X,i} - r_i)X_i\left(\frac{\partial\hat{V}}{\partial X_i} + \phi_i + \psi_{B,i} + \sum_j\delta_{F,i,j}P_{F,i,j}^{-}\right)\\
&+ \sum_{i,j}\delta_{F,i,j}r_{F,i,j}X_iP_{F,i,j}^{-} + \sum_i r_{\phi,i}X_i\phi_i + \sum_i r_{\psi,i}X_i\psi_i\\
&+ \left(\frac{r_C-\gamma_C}{1-R_C}\right)(g_C - \hat{V})\Big\}dt + \Big\{g_B - \hat{V} - \sum_{i,j}(1-R_{F,i,j})X_i\delta_{F,i,j}P_{F,i,j}^{-}\Big\}dJ_B
\end{aligned}
\tag{32}
$$

In order to enhance the aesthetic appeal of the final result we introduce some further notation

$$
\begin{aligned}
\mathcal{A}_S &:= \sum_i (\gamma_{S,i} - q_{S,i} - \rho_{S_i,X_i}\sigma_{S,i}(t,S_i)\sigma_{X,i}(t,X_i))S_i\frac{\partial}{\partial S_i} \\
\mathcal{A}_X &:= \sum_i (\gamma_{X,i} - r_i)X_i\frac{\partial}{\partial X_i} \\
\mathcal{A}_r &:= \sum_i (a_{r,i}(t,r_i) - \rho_{X_i,r_i}\sigma_{r,i}(t,r_i)\sigma_{X,i}(t,X_i))\frac{\partial}{\partial r_i} \\
\mathcal{A} &:= \mathcal{A}_2 + \mathcal{A}_S + \mathcal{A}_r + \mathcal{A}_X \\
\lambda_C &:= \frac{r_C - \gamma_C}{1 - R_C} \\
\gamma_{F,i,j} &:= r_{F,i,j} + \gamma_{X,i} - r_i \\
\gamma_{\phi,i} &:= r_{\phi,i} + \gamma_{X,i} - r_i \\
\gamma_{\psi,i} &:= r_{\psi,i} + \gamma_{X,i} - r_i.
\end{aligned}
$$

We are now in a position to write down the final version of the balance sheet dynamics as

$$
\begin{aligned}
d(\hat{V} + \Pi) = \Big\{ &\frac{\partial \hat{V}}{\partial t} + \mathcal{A}\hat{V} + \lambda_C(g_C - \hat{V}) + \sum_{i,j}\gamma_{F,i,j}\delta_{F,i,j}X_iP^-_{F,i,j} \\
&+ \sum_i \gamma_{\phi,i}X_i\phi_i + \sum_i \gamma_{\psi,i}X_i\psi_{B,i}\Big\}dt \\
&+ \Big\{g_B - \hat{V} - \sum_{i,j}(1 - R_{F,i,j})X_i\delta_{F,i,j}P^-_{F,i,j}\Big\}dJ_B.
\end{aligned} \tag{33}
$$

The no-basis condition (4) implies that $\gamma_{F,i,j} = \gamma_{X,i} + (1 - R_{F,i,j})\lambda_B$ which we insert into (33) to obtain

$$
\begin{aligned}
d(\hat{V} + \Pi) = \Big\{ &\frac{\partial \hat{V}}{\partial t} + \mathcal{A}\hat{V} + \lambda_B(g_B - \hat{V}) + \lambda_C(g_C - \hat{V}) + \sum_{i,j}\gamma_{X,i}\delta_{F,i,j}X_iP^-_{F,i,j} \\
&+ \sum_i \gamma_{\phi,i}X_i\phi_i + \sum_i \gamma_{\psi,i}X_i\psi_{B,i}\Big\}dt \\
&+ \epsilon_h(dJ_B - \lambda_B dt).
\end{aligned} \tag{34}
$$

where we also defined the balance sheet hedge error at bank default as $\epsilon_h := g_B - \hat{V} - \sum_{i,j}(1 - R_{F,i,j})X_i\delta_{F,i,j}P^-_{F,i,j}$.

# B Derivation of the valuation adjustments

In this section we derive the shareholder and firm value adjustments. Since they apply strictly before the default of any of the parties we write $P^-_{F,i,j} = P_{F,i,j}$ from now on. Variation and initial margin collateral allow multiple eligible currencies and has full substitution rights.

## B.1 Shareholder value adjustments

We first insert the single bond strategy $\delta_{F,i_F} X_{i_F} P_{F,i_F} = -\hat{V} - \phi^d - \psi^d_B$ into the balance sheet dynamics (33) before setting the $dt-$terms to zero to obtain

$$\begin{aligned} \frac{\partial \hat{V}}{\partial t} + \mathcal{A}\hat{V} - (\gamma_F + \lambda_C)\hat{V} &= -\lambda_C g_C + (\gamma_F - \gamma_{\phi,E})\phi^d + (\gamma_F - \gamma_{\psi,E})\psi^d_B \\ \hat{V}(T, r, S, X, 0, 0) &= H(r, S, X). \end{aligned} \tag{35}$$

The partial differential equation (35) is expressed directly in terms of the effective funding rate $\gamma_F$ introduced in Section 5.1 rather than $\lambda_B$. Next we insert the ansatz $\hat{V} = V^d_\phi + U$ into (35) and subtract $\gamma_{\phi,E} V^d_\phi$ from both sides to obtain

$$\begin{aligned} \frac{\partial V^d_\phi}{\partial t} + \mathcal{A}V^d_\phi - (\gamma_F + \lambda_C)V^d_\phi - \gamma_{\phi,E}V^d_\phi + \frac{\partial U}{\partial t} + \mathcal{A}U - (\gamma_F + \lambda_C)U &= -\gamma_{\phi,E}V^d_\phi - \lambda_C g_C + (\gamma_F - \gamma_\phi)\phi^d \\ &\quad + (\gamma_F - \gamma_{\psi,E})\psi^d_B. \\ V^d_\phi(T, r, S, X) + U(T, r, S, X) &= H(r, S, X). \end{aligned}$$

Recognising the PDE (14) with $\xi = \gamma_{\phi,E}$ satisfied by $V^d_\phi$ on the left hand side allows us to eliminate terms, and after some algebra we get

$$\begin{aligned} \frac{\partial U}{\partial t} + \mathcal{A}U - (\gamma_F + \lambda_C)U &= \lambda_C(V^d_\phi - g_C) + (\gamma_F - \gamma_{\phi,E})(V^d_\phi + \phi^d) + (\gamma_F - \gamma_{\psi,E})\psi^d_B \\ U(T, r, S, X) &= 0. \end{aligned} \tag{36}$$

Finally the Feynman-Kac theorem gives the solution $U(t) = FCVA(t) + FVA(t) + MVA(t)$ where the valuation adjustments are given in (17) to (19).

The integral formula for $COLVA(t)$ is proved in a similar fashion by inserting the ansatz $V^d_r(t) = V^d_\phi(t) - COLVA(t)$ into the PDE (14) with $\xi = \gamma_{X,i_r}$.

### B.2 Firm value adjustments

By Section 4.2 the firm value is given as the solution to the shareholder value PDE (35) with $\epsilon_h = 0$. If we furthermore invoke the zero basis condition (4) then

$$\begin{aligned} \frac{\partial \hat{V}_{\mathrm{FV}}}{\partial t} + \mathcal{A}\hat{V}_{\mathrm{FV}} - (\gamma_{X,i_F} + \lambda_B + \lambda_C)\hat{V}_{\mathrm{FV}} &= -\lambda_B g_B - \lambda_C g_C \\ &\quad - (\gamma_{\phi,E} - \gamma_{X,i_F})\phi - (\gamma_{\psi,E} - \gamma_{X,i_F})\psi_B \\ \hat{V}_{\mathrm{FV}}(T,r,S,X,0,0) &= H(r,S,X). \end{aligned} \tag{37}$$

The process of deriving the firm value adjustments is very similar to that of the shareholder value adjustments in Section B.1. First the ansatz $\hat{V}_{\mathrm{FV}} = V_\phi^d + U_{\mathrm{FV}}$ is inserted into (37). Second $\gamma_{\phi,E}V_\phi^d$ is subtracted from both sides of the PDE and the PDE (14) with $\xi = \gamma_{\phi,E}$ is used to eliminate terms. This yields

$$\begin{aligned} \frac{\partial U_{\mathrm{FV}}}{\partial t} + \mathcal{A}U_{\mathrm{FV}} - (\gamma_{X,i_F} + \lambda_B + \lambda_C)U_{\mathrm{FV}} &= \lambda_B(V_\phi^d - g_B) + \lambda_C(V_\phi^d - g_C) \\ &\quad - (\gamma_{\phi,E} - \gamma_{X,i_F})(V_\phi^d + \phi^d) - (\gamma_{\psi,E} - \gamma_{X,i_F})\psi_B^d \\ U_{\mathrm{FV}}(T,r,S,X) &= 0 \end{aligned} \tag{38}$$

so by the Feynman-Kac theorem $U_{\mathrm{FV}} = FTDCVA + FTDCVA + FVMVVA + FVIMVA$ where the valuation adjustments are given in (20) to (23).

## C Multiple netting sets and CSAs

Until now we have assumed that all the trades belong to a single netting set and CSA. Like in the single currency economy in Kjaer [3] we let $k,l$ denote the credit support annex and netting set indices, respectively. Each CSA has effective curves $\gamma_{\phi,E,k}$ and $\gamma_{\psi,E,k}$ for variation and initial margin. As in the single currency model the closeout process is done per netting set and the proceeds are additive so it holds that $g_B = \sum_l g_{B,l}$ and $g_C = \sum_l g_{C,l}$. Applying the semi-replication arguments of Section A to these boundary conditions and give the shareholder value as

$$\hat{V} = \sum_k V_{r,k}^d + \sum_k COLVA_k + \sum_l FCVA_l + \sum_k FVA_k + \sum_k MVA_k$$

with $V_{r,k}^d$ being the OIS discounted value per CSA which uses the curve $\gamma_{X,i_{r,k}}$ and

$$\begin{aligned}
COLVA_k &= -\mathbb{E}_t \left[ \int_t^T (\gamma_{\phi,E,k}(u) - \gamma_{X,i_{r,k}}(u)) D_{\gamma_{X,i_{r,k}}}(t,u) V^d_{\phi,k}(u) du \right] \\
FCVA_l &= -\mathbb{E}_t \left[ \int_t^T \lambda_C(u) D_{\gamma_F+\lambda_C}(t,u)(V^d_{\phi,l}(u) - g_{C,l}(u)) du \right] \\
FVA_k &= -\mathbb{E}_t \left[ \int_t^T (\gamma_F(u) - \gamma_{\phi,E,k}(u)) D_{\gamma_F+\lambda_C}(t,u)(V^d_{\phi,k}(u) + \phi^d_k(u)) du \right] \\
MVA_k &= -\mathbb{E}_t \left[ \int_t^T (\gamma_F(u) - \gamma_{\psi,E,k}(u)) D_{\gamma_F+\lambda_C}(t,u) \psi^d_{B,k}(u) du \right].
\end{aligned}$$

Similarly the firm value is given by

$$\hat{V}_{\mathrm{FV}} = \sum_k V^d_{r,k} + \sum_k COLVA_k + \sum_l FTDCVA_l + \sum_l FTDDVA_l + \sum_k FVVMVA_k + \sum_k FVIMVA_k$$

with $COLVA_k$ given above and

$$\begin{aligned}
FTDCVA_l &= -\mathbb{E}_t \left[ \int_t^T \lambda_C(u) D_{\gamma_{X,i_F}+\lambda_B+\lambda_C}(t,u)(V^d_{\phi,l}(u) - g_{C,l}(u)) du \right] \\
FTDDVA_l &= -\mathbb{E}_t \left[ \int_t^T \lambda_B(u) D_{\gamma_{X,i_F}+\lambda_B+\lambda_C}(t,u)(V^d_{\phi,l}(u) - g_{B,l}(u)) du \right] \\
FVVMVA_k &= -\mathbb{E}_t \left[ \int_t^T (\gamma_{X,i_F}(u) - \gamma_{\phi,E,k}(u)) D_{\gamma_{X,i_F}+\lambda_B+\lambda_C}(t,u)(V^d_{\phi,k}(u) + \phi^d_k(u)) du \right] \\
FVIMVA_k &= -\mathbb{E}_t \left[ \int_t^T (\gamma_{X,i_F}(u) - \gamma_{\psi,E,k}(u)) D_{\gamma_{X,i_F}+\lambda_B+\lambda_C}(t,u) \psi^d_{B,k}(u) du \right].
\end{aligned}$$

# D Reporting currency invariance

In this section we embed a single currency $i$ economy in our multi-currency economy such that (a) $H(r,S,X) = X_i H_i(r_i, S_i)$, (b) $i_F = i_C = i$, (c) $\gamma_{\phi,E} = \gamma_{\phi,i}$ and $\gamma_{\psi,E} = \gamma_{\psi,i}$ (d) $g_B = X_i g_{B,i}$ and (e) $g_C = X_i g_{C,i}$. From this setup it also follows that $\phi^d = X_i \phi_i$ and $\psi^d_B = X_i \psi_{B,i}$. Our aim is to insert the assumptions (a) to (d) and the ansatz $\hat{V}(t,r,S,X,0,0) = X_i(t)\hat{V}_i(t,r_i,S_i,0,0)$ into the shareholder value PDE (12) and simplify. First, assumptions (a) to (d) combined with the single currency $i$ funding strategy yields a hedge error of the form $\epsilon_h = X_i \epsilon_{h,i}$ with $\epsilon_{h,i} = g_{B,i} - \hat{V}_i - (1 - R_{F,i})\delta_{F,i} P_{F,i}$. Second, it is straightforward but somewhat tedious to show that $\mathcal{A}(X_i \hat{V}_i) = X_i \mathcal{A}_i \hat{V}_i + (\gamma_{X,i} - r_i) X_i \hat{V}_i$ with the single currency operator $\mathcal{A}_i$ defined by

$$\begin{aligned}
\mathcal{A}_i = {} & \frac{1}{2}\sigma^2_{S,i}(t,S_i) S^2 \frac{\partial^2}{\partial S_i^2} + \frac{1}{2}\sigma^2_{r,i}(t,r_i) \frac{\partial^2}{\partial r_i^2} + \rho_{S_i,r_i} \sigma_{r,i}(t,r_i) \sigma_{S,i}(t,S) S_i \frac{\partial^2}{\partial S_i \partial r_i} \\
& + (\gamma_{S,i} - q_{S,i}) S_i \frac{\partial}{\partial S_i} + a_{r,i}(t,r_i) \frac{\partial}{\partial r_i}.
\end{aligned}$$

Third, the funding strategy implies that $\sum_{i,j} s_{X,i}\delta_{F,i,j}X_i P_{F,i,j} = -X_i(\gamma_{X,i} - r_d)(\hat{V}_i + \phi_i + \psi_{B,i})$. Inserting what we have got so far into the PDE (12) and re-arranging some terms yield

$$\begin{aligned} X_i \frac{\partial \hat{V}_i}{\partial t} + X_i \mathcal{A}_i \hat{V}_i - X_i r_i \hat{V}_i + X_i(\gamma_{X,i} - r_d)\hat{V}_i - X_i(\lambda_B + \lambda_C)\hat{V}_i &= -X_i \lambda_B g_{B,i} - X_i \lambda_C g_{C,i} \\ &\quad - X_i(r_{\phi,i} - r_i)\phi_i - X_i(r_{\psi,i} - r_i)X_i\psi_{B,i} \\ &\quad + X_i(\gamma_{X,i} - r_d)\hat{V}_i + X_i \lambda_B \epsilon_{h,i} \\ X_i \hat{V}_i(T, r_i, S_i, 0, 0) &= X_i H_i(r_i, S_i). \end{aligned}$$

Next we can divide both sides by $X_i$ so it is clear that $\hat{V}_i$ is the solution to the partial differential equation

$$\begin{aligned} \frac{\partial \hat{V}_i}{\partial t} + \mathcal{A}_i \hat{V}_i - (r_i + \lambda_B + \lambda_C)\hat{V}_i &= -\lambda_B g_{B,i} - \lambda_C g_{C,i} - (r_{\phi,i} - r_i)\phi_i \\ &\quad - (r_{\psi,i} - r_i)\psi_{B,i} + \lambda_B \epsilon_{h,i} \\ \hat{V}_i(T, r_i, S_i, 0, 0) &= H_i(r_i, S_i). \end{aligned} \tag{39}$$

We recognise that (39) is identical to the single currency economy shareholder value differential equation in Kjaer [3] so $\hat{V}_i$ may be decomposed into single currency valuation adjustments. By setting $\epsilon_{h,i} = 0$ we can do the same for the firm value $\hat{V}_{\text{FV}} = X_i \hat{V}_{\text{FV},i}$.